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Design of Neural Network With Levenberg-Marquardt and Bayesian Regularization Backpropagation for Solving Pantograph Delay Differential Equations

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ABSTRACT In this paper, novel computing paradigm by exploiting the strength of feed-forward artificial neural networks (ANNs) with Levenberg-Marquardt Method (LMM), and Bayesian Regularization Method (BRM) based backpropagation is presented to find the solutions of initial value problems (IVBs) of linear/nonlinear pantograph delay differential equations (LP/NP-DDEs). The dataset for training, testing and validation is created with reference to known standard solutions of LP/NP-DDEs. ANNs are implemented using the said dataset for approximate modeling of the system on mean squared error based merit functions, while learning of the adjustable parameters is conducted with efficacy of LMM (ANN-LMM) and BRMs (ANN-BRM). The performance of the designed algorithms ANN-LMM and ANN-BRM on IVPs of first, second and third order NP-FDEs are verified by attaining a good agreement with the available solutions having accuracy in the range from 10^{-5} to 10^{-8} and are further endorsed through error histograms and regression measures.

INDEX TERMS Artificial neural networks, Levenberg-Marquardt method, Bayesian regularization method, nonlinear pantograph equation, regression analysis, intelligent computing, numerical computing.

I. INTRODUCTION

A particular form of functional differential equations that involve proportional delays are called pantograph or generalized pantograph equations. The word 'pantograph' has been introduced by Ockendon and Tayler (1971) [1] in a project (collection of current by pantograph head of an electric locomotive system). The general form of first, second and third order functional differential equations of the pantograph type are:

$$\frac{df}{dx} = y(f(x), f(h(x)), x), \quad f(0) = c_1 \tag{1}$$

$$\frac{d^2 f}{dx^2} = y(f(x), \quad \frac{df}{dx}, f(h(x)), x),$$
$$f(0) = c_1, \quad \frac{d}{dx} f(0) = c_2 \tag{2}$$

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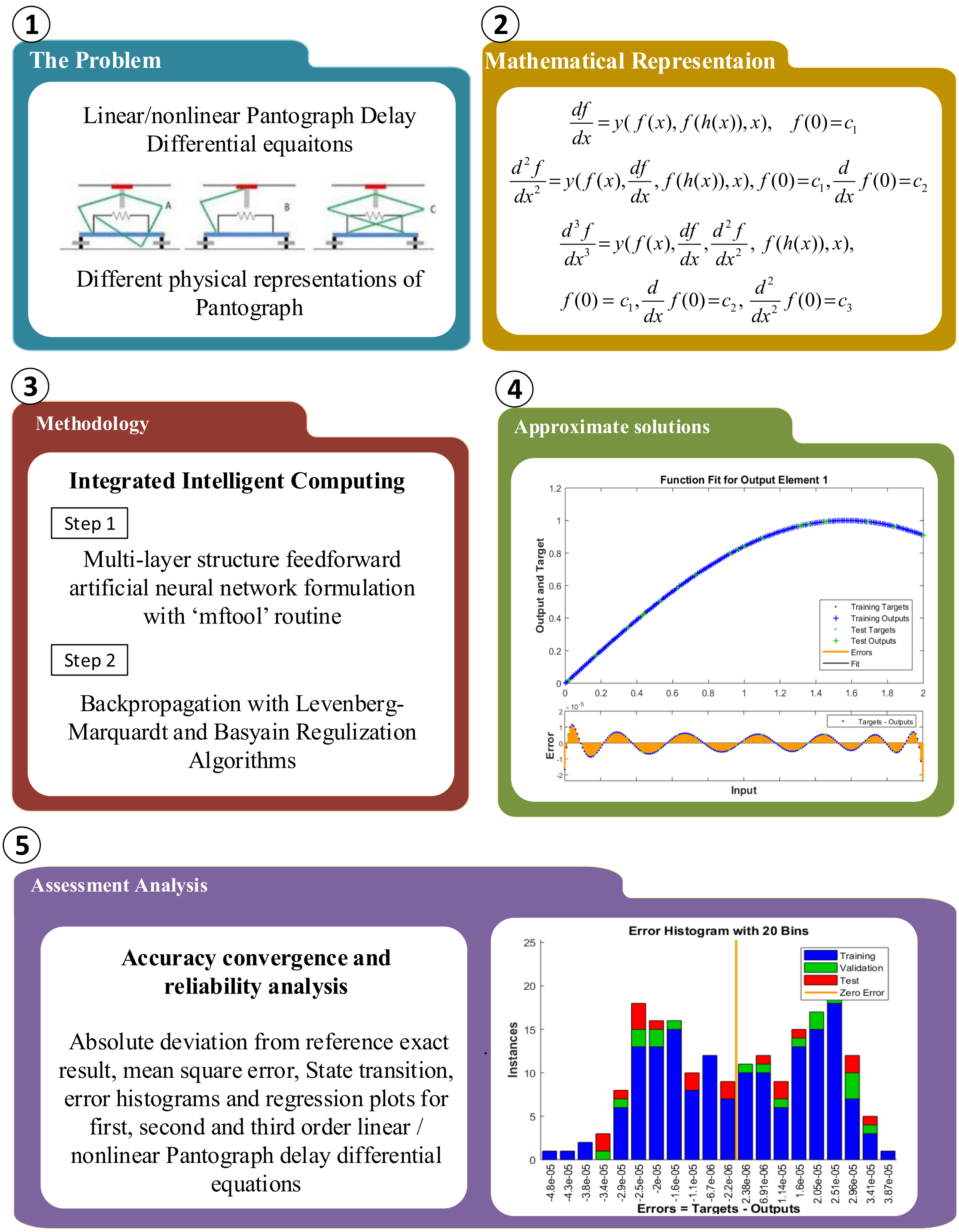

FIGURE 1. Block structure representation of workflow of the system.

$$\frac{d^3f}{dx^3} = y(f(x), \quad \frac{df}{dx}, \frac{d^2f}{dx^2}, f(h(x)), x),$$
$$f(0) = c_1, \quad \frac{d}{dx}f(0) = c_2, \quad \frac{d^2}{dx^2}f(0) = c_3 \tag{3}$$

The said equations have applicability in diverse range of subject areas, for instance, coherent states in quantum theory [2], control system [3] and cell-growth modeling in biology [4]. According to recent literature available, a number of numerical solvers have shown a great potential to solve Pantograph functional differential equations using different methods such as collection method [5], spectral tau method [6], Chebyshev spectral methods [7], multi-stage optimal homotopy asymptotic method [8], orthonormal

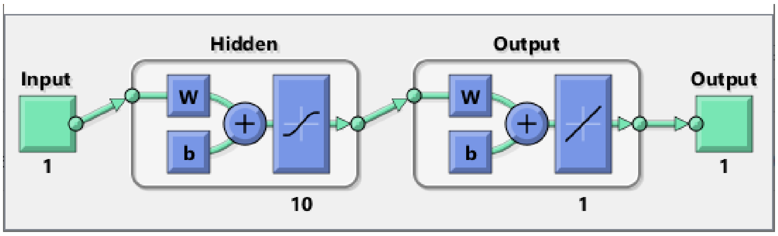

FIGURE 2. Neural network model for selected architecture.

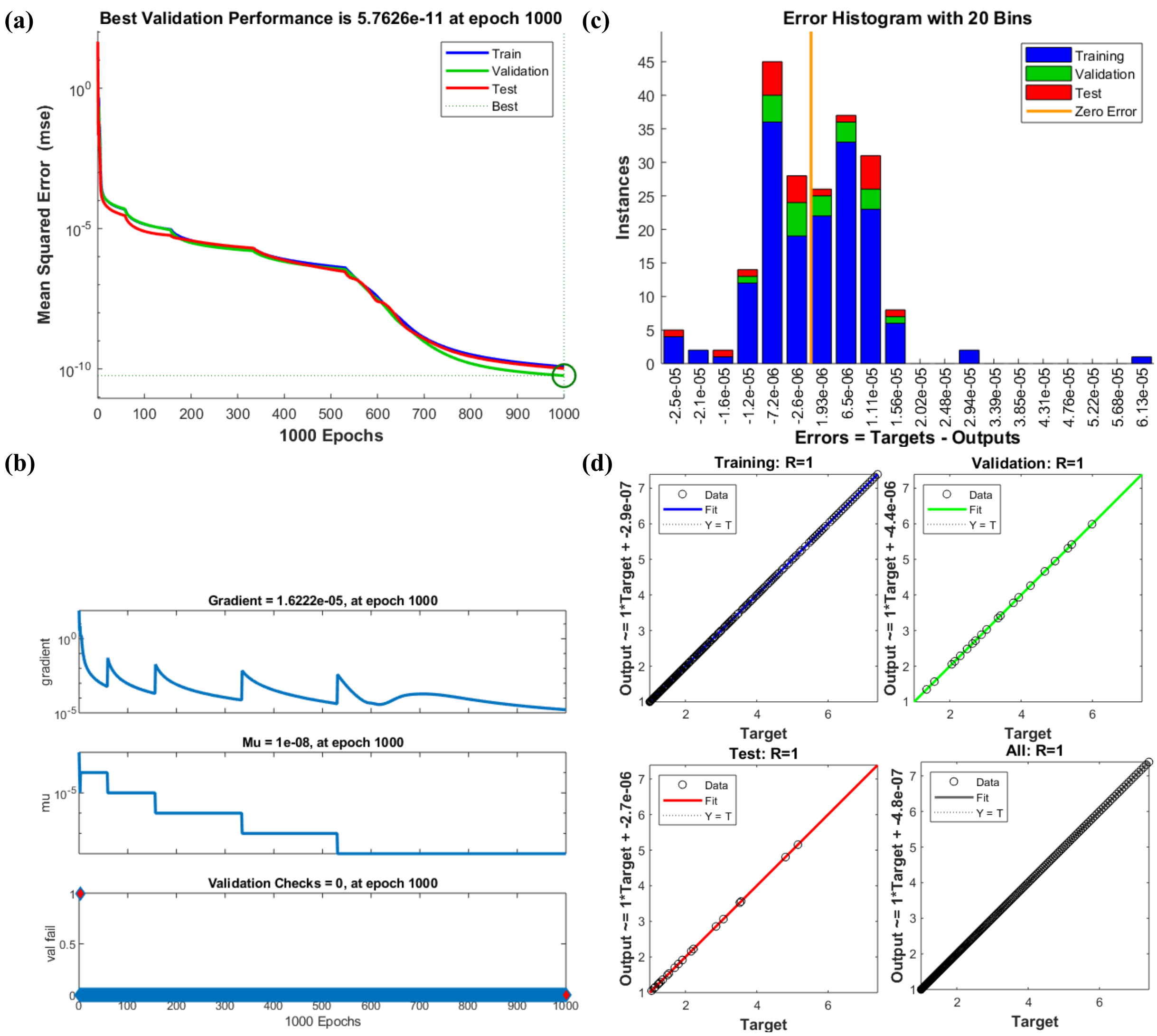

FIGURE 3. Execution of ANN-LMM for solving LP-DDE in problem 4.1 (a) performance plots (b) Training state parameters (c) Error Histogram (d) Regression plots.

Bernstein polynomials method [9], Adomian decomposition method [10], Genocchi operational matrix approach [11], least squares-Epsilon-Ritz method [12], collocation approach through first Boubeker polynomials [13], Taylor operation method [14], Taylor wavelets method [15], multi-wavelets Galerkin method [16], modified Chebyshev collocation method [17], Sinc numerical method [18], Müntz-Legendre wavelet operational matrix approach [19], Legendre–Gauss quadrature rule based scheme [20], Laplace transform method [21], multistep block method [22], computational Legendre Tau method [23], fully-geometric mesh one-leg methods [24], Euler–Maruyama method [25] and so on. In all of these methods, the deterministic solution is generally given in different forms with stability and convergence analysis, while the outcomes show that all of these methods have their own limitations and advantages in comparison

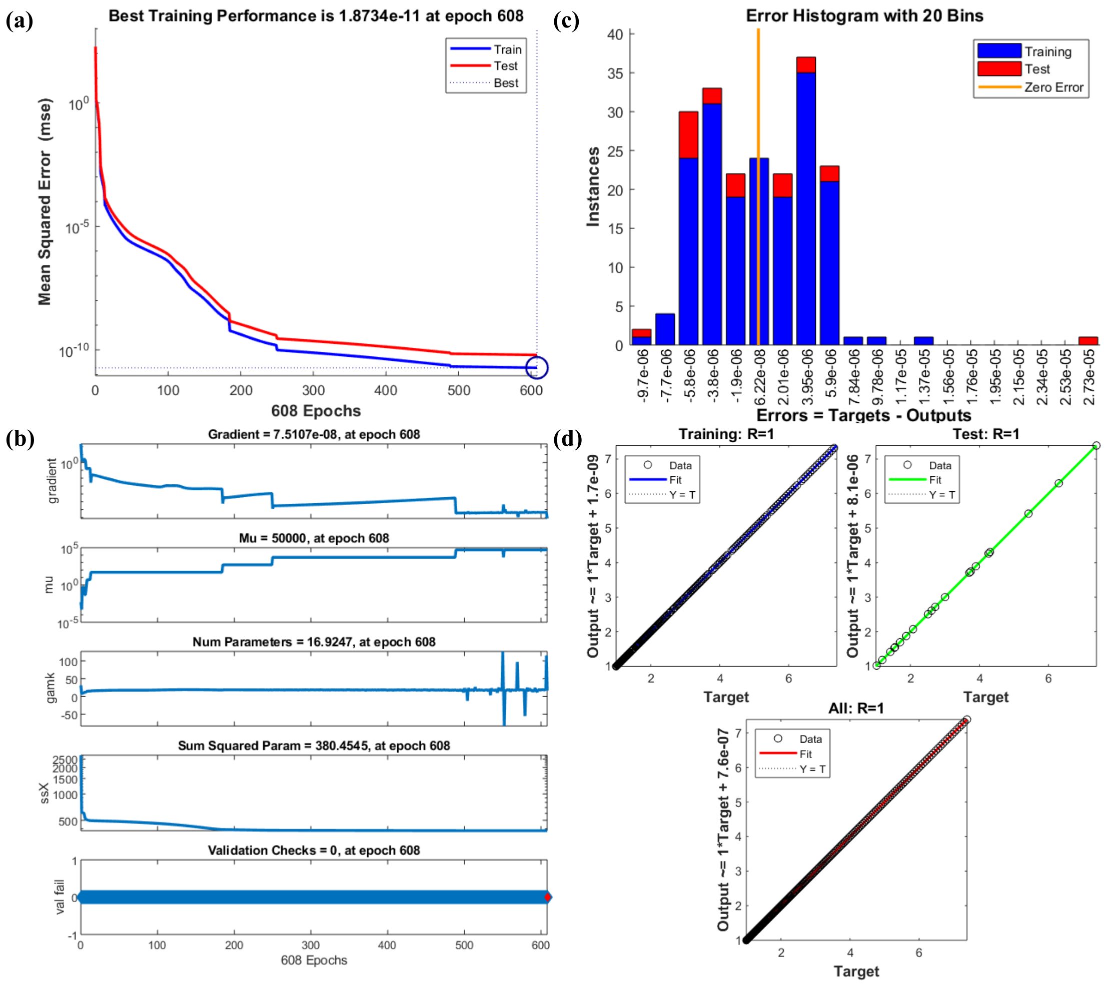

FIGURE 4. Execution of ANN-BRM for solving LP-DDE in problem 4.1 (a) performance plots (b) Training state parameters (c) Error Histogram (d) Regression plots.

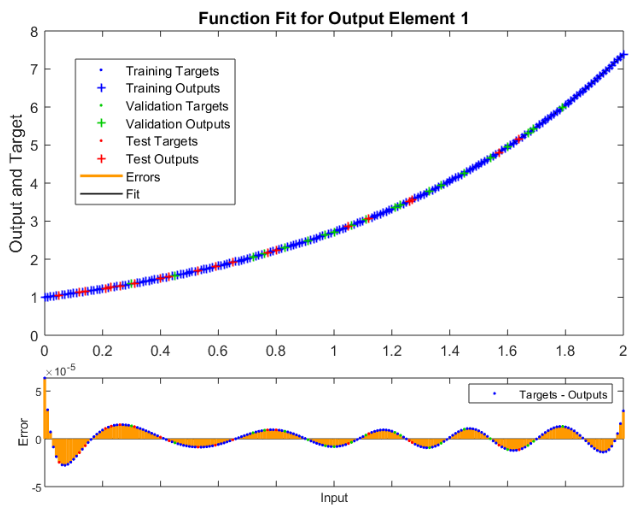

FIGURE 5. Approximate solution with error analysis for ANN-LMM for problem 4.1.

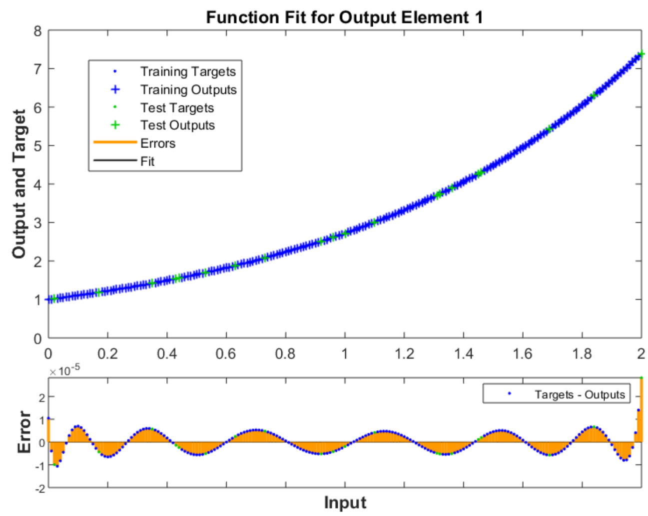

FIGURE 6. Approximate solution with error analysis for ANN-BRM for problem 4.1.

to others in certain applications. However, artificial intelligence (AI) based numerical solvers using soft computing or machine learning approaches look promising to be further researched for nonlinear Pantograph differential systems.

The AI based numerical approaches have been used broadly for solution of differential equations arising in

TABLE 1. Results of ANN-LMM for solving linear and nonlinear Pantograph delay differential equations.

LP/NP-DDE Problem	Mean square error			Performance	Gradient	Mu	Epoch	Time
	Training	Validation	Testing					
4.1	3.46E-10	4.41E-10	6.10E-10	3.46E-10	7.51E-08	1.00E-07	1000	3
4.2	2.08E-04	2.72E-04	1.02E-04	1.02E-04	3.68E-04	1.00E-08	22	< 0.5
4.3	1.36E-09	1.31E-09	1.32E-09	5.35E-10	2.02E-06	1.00E-10	1000	1
4.4	1.33E-10	2.49E-10	1.29E-10	1.28E-10	2.32E-07	1.00E-08	1000	2
4.5	2.10E-04	1.10E-04	2.39E-04	2.61E-04	1.08E-06	1.00E-09	191	< 0.5

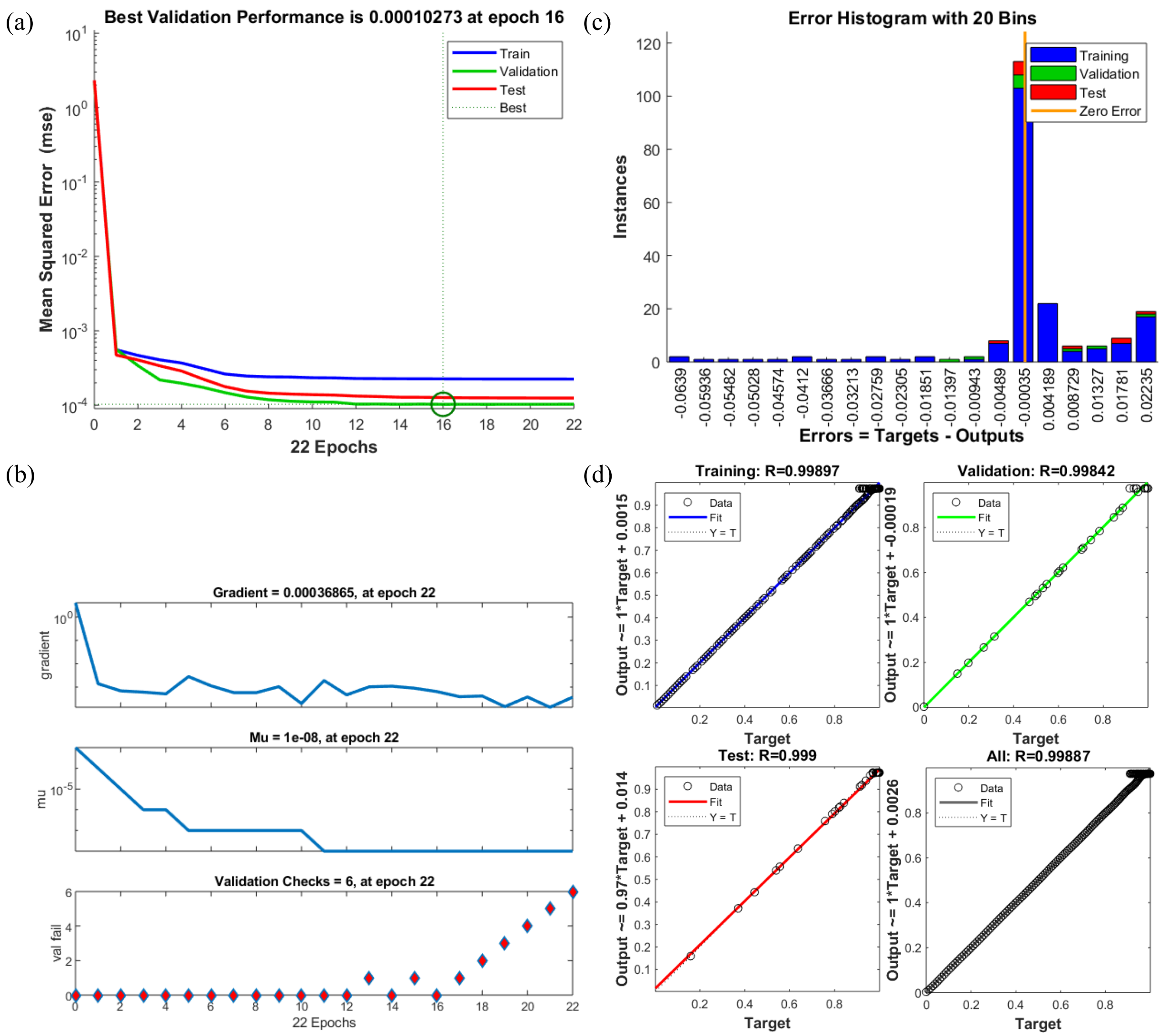

FIGURE 7. Execution of ANN-LMM for solving NP-DDE in problem 4.2 (a) performance plots (b) Training state parameters (c) Error Histogram (d) Regression plots.

diffrenet applications [26]–[31]. A few recent studies of paramount significance reported include Van-der-Pol oscillatory nonlinear systems [32], [33], solution of mathematical model in nonlinear optics [34], models of electrically conducting solids [35], [36], analysis of nonlinear reactive transport model [37], fuel ignition model in combustion theory [38], Jeffery Hamel flow models [39], [40], thin film flow models [41], [42], mathematical models involving Carbon nanotubes [43], [44], astrophysics [45], [46], nonlinear circuit theory models [47], [48], dusty plasma [49], [50], atomic physics [51], [52], heartbeat dynamic models [53], HIV infection spread models [54], [55], piezoelectric

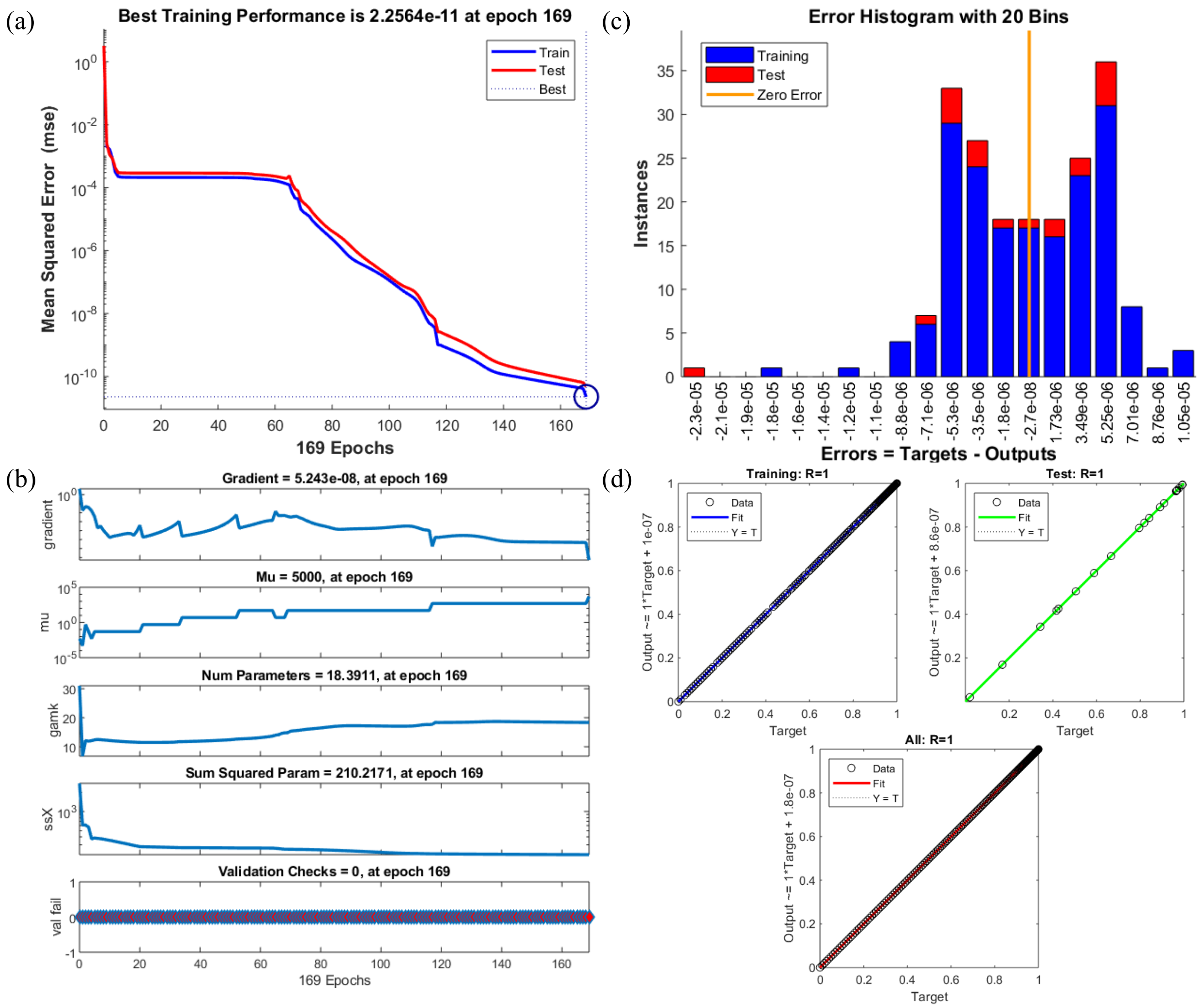

FIGURE 8. Execution of ANN-BRM for solving NP-DDE in problem 4.2 (a) performance plots (b) Training state parameters (c) Error Histogram (d) Regression plots.

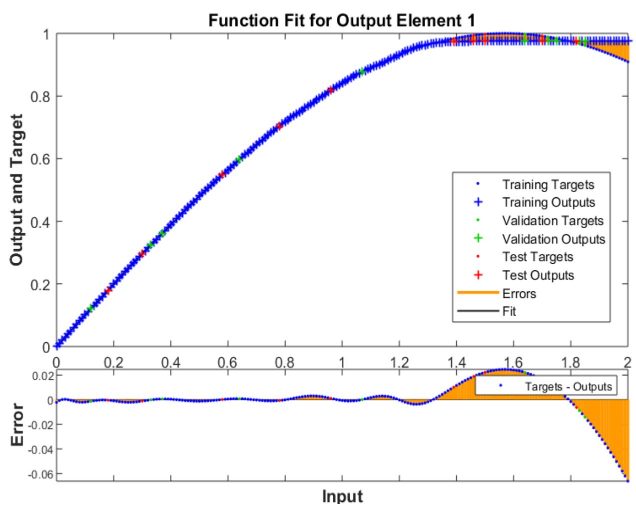

FIGURE 9. Approximate solution with error analysis for ANN-LMM for problem 4.2.

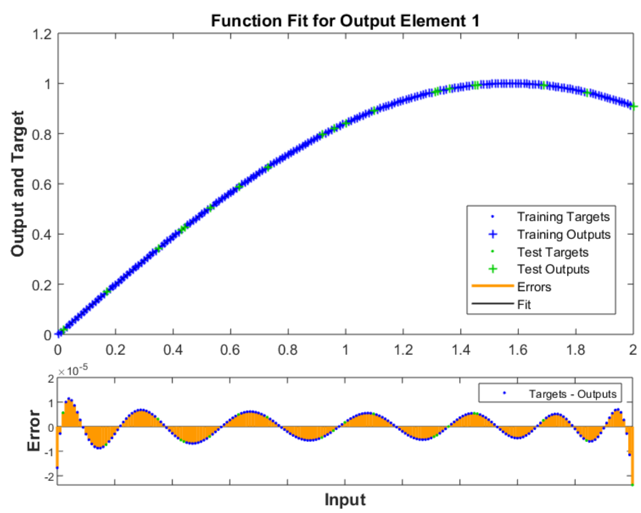

FIGURE 10. Approximate solution with error analysis for ANN-BRM for problem 4.2.

transducer modeling [56], energy [57], [58], wind power [59], [60], and financial models [61], [62]. Beside these stiff nonlinear fractional dynamic modeling with Riccati fractional differential equations (FrDEs) [63], [64] and Bagley-Torvik FrDEs [65] are other potential applications of AI based algorithms. As per our literature survey no researcher yet has

TABLE 2. Results of ANN-BRM for solving linear and nonlinear Pantograph delay differential equations.

P/NP-DDE Problem	Mean square error			Performance	Gradient	Mu	Epoch	Time
	Training	Validation	Testing					
4.1	8.46E-12	0.00	1.06E-11	1.87E-11	7.51E-08	5.00E+04	608	0.03
4.2	2.12E-11	0.00	1.82E-11	2.25E-11	5.24E-08	5.00E+03	169	0.01
4.3	3.22E-13	0.00	3.08E-13	4.98E-10	1.38E-07	5.00E+03	1000	0.01
4.4	2.60E-13	0.00	3.18E-13	2.61E-13	9.75E-08	5.00E+03	577	0.01
4.5	2.29E-13	0.00	1.63E-13	2.30E-13	2.02E-09	5.00E+04	746	0.02

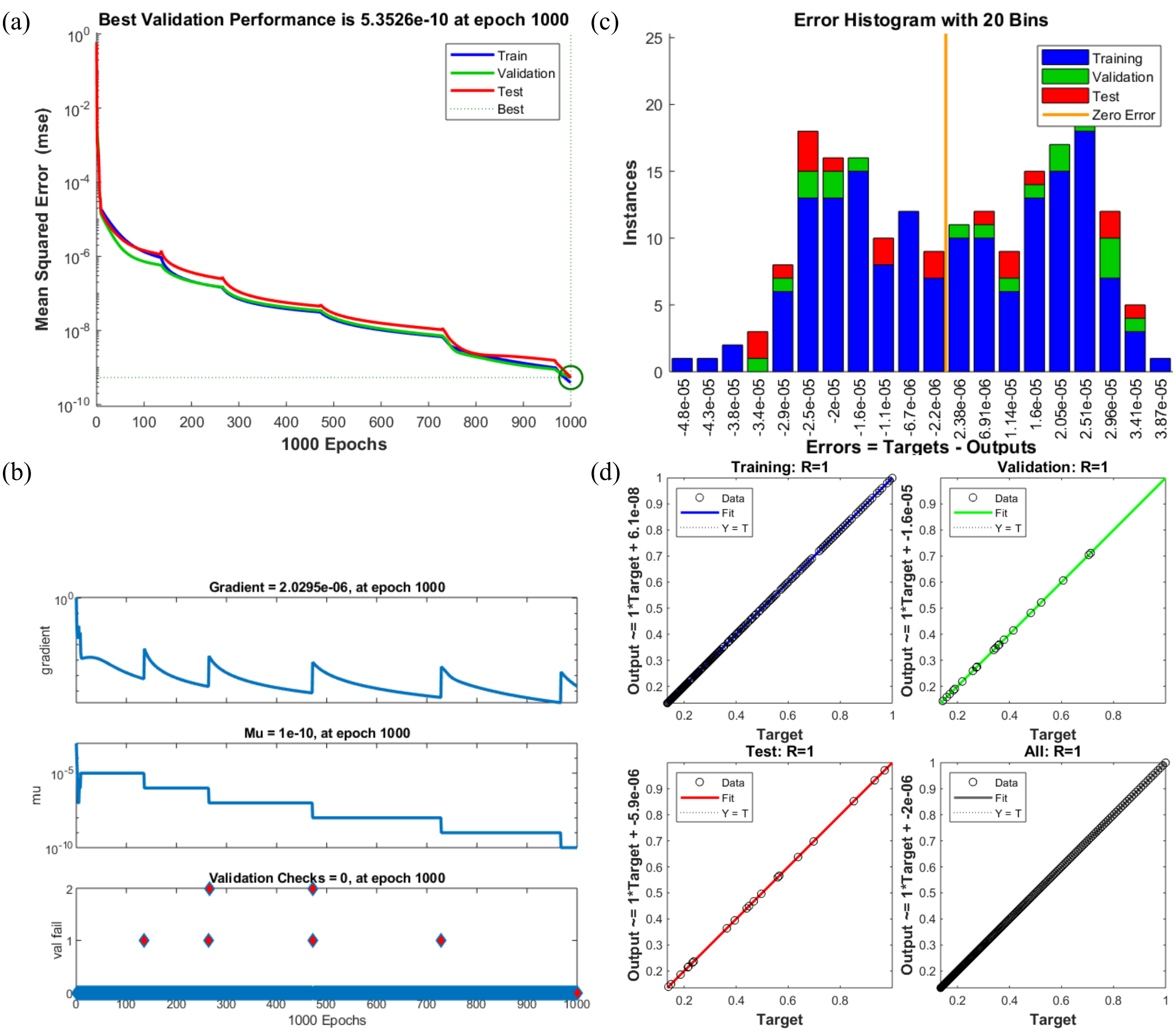

FIGURE 11. Execution of ANN-LMM for solving LP-DDE in problem 4.3 (a) performance plots (b) Training state parameters (c) Error Histogram (d) Regression plots.

applied AI techniques through Levenberg-Marquardt Method (LMM) and Bayesian Regularization Method (BRM) based backpropagation of neural networks to solve initial value problems (IVBs) of linear/nonlinear pantograph delay differential equations (LP/NP-DDEs). This encourages the authors to investigate an AI technique to solve IVPs of LP/NP-DDEs represented in equations (1-3). The innovative aspects of the proposed computing platform are highlighted as follows:

- A novel design of two-layer feed-forward artificial neural networks (ANNs) backpropagated with Levenberg-Marquardt method (LMM), i.e., ANN-LMM and Bayesian Regularization Method (BRM), i.e., ANN-BRM is presented for finding the solution of IVBs of LP/NP DDEs.
- The mean squared error based merit function is constructed for implementation of both ANN-LMM and ANN-BRM for approximate modeling of the

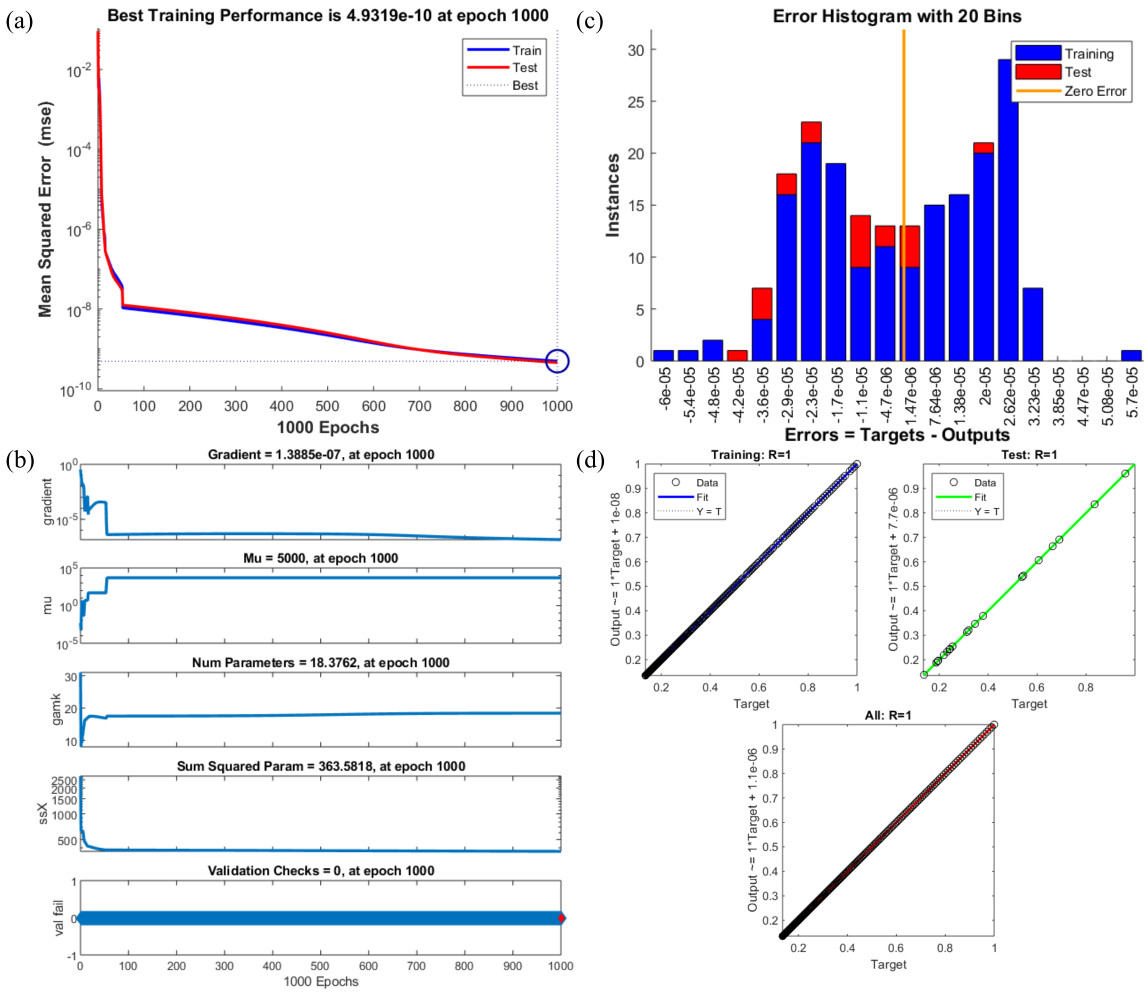

FIGURE 12. Execution of ANN-BRM for solving LP-DDE in problem 4.3 (a) performance plots (b) Training state parameters (c) Error Histogram (d) Regression plots.

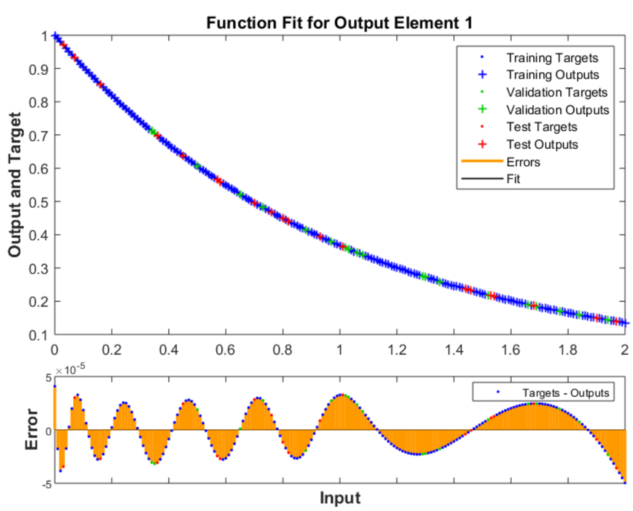

FIGURE 13. Approximate solution with error analysis for ANN-LMM for problem 4.3.

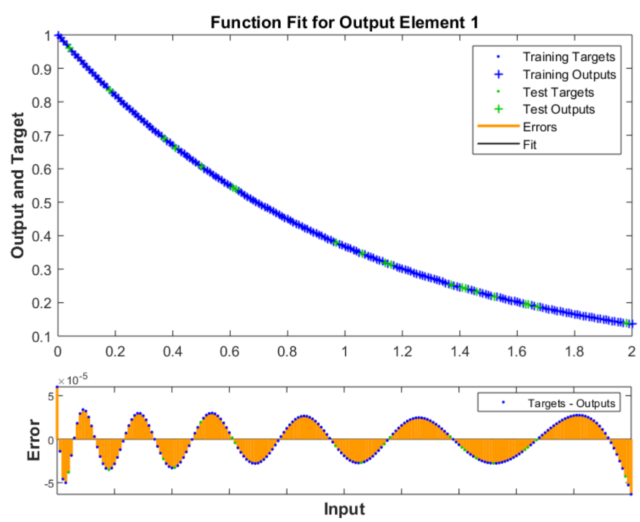

FIGURE 14. Approximate solution with error analysis for ANN-BRM for problem 4.3.

LP/NP-DDEs through reference dataset for training, testing and validation

- Learning of the decision variables of neural network to optimize the merit function at each epoch is conducted with efficacy of backpropagation with Levenberg-Marquardt and Bayesian regularization.
- Accurate, reliable and convergent performance of the designed schemes ANN-LMM and ANN-BRM is

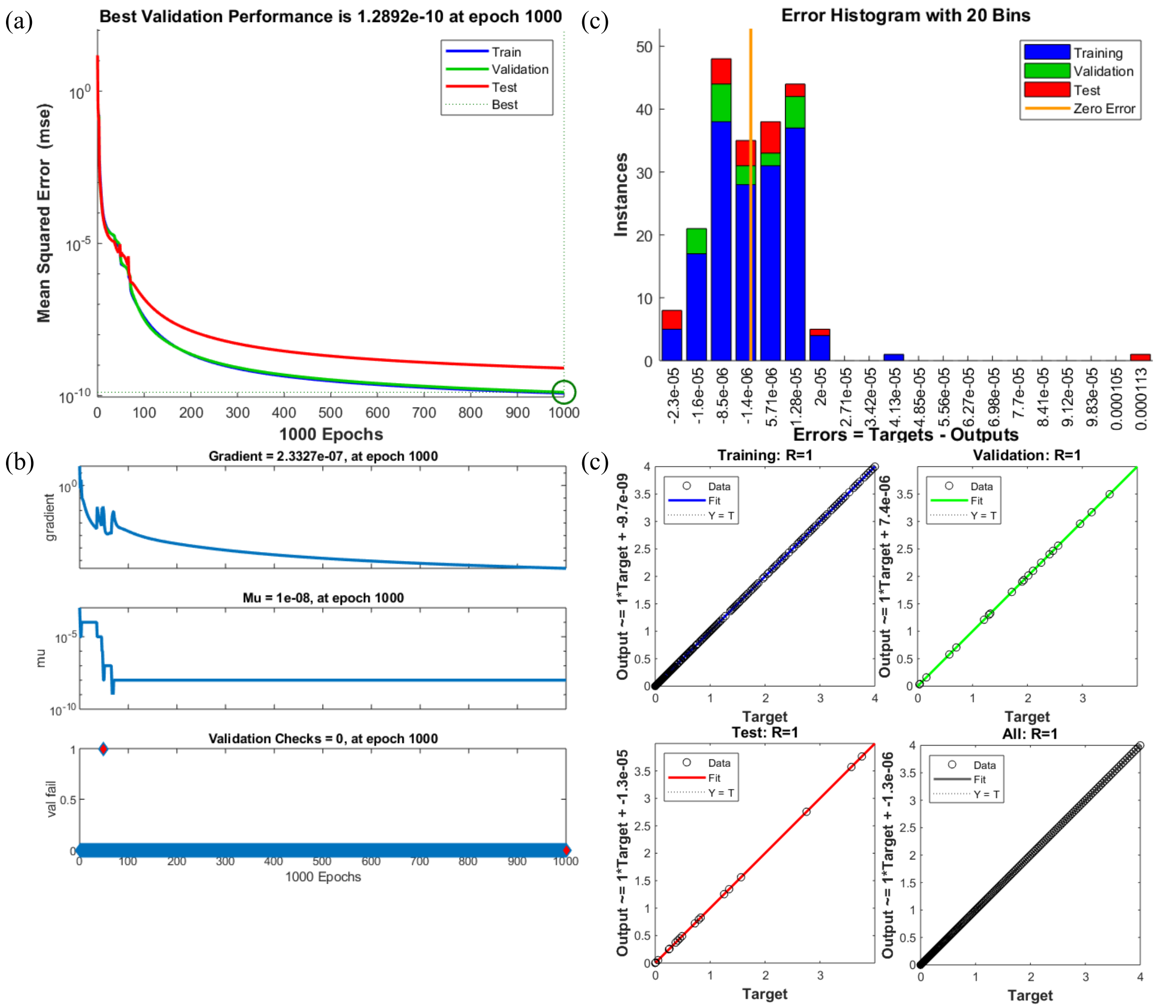

FIGURE 15. **Execution of ANN-LMM for solving LP-DDE in problem 4.4 (a) performance plots (b) Training state parameters (c) Error Histogram (d) Regression plots.**

substantiated for first, second and third order variant LP/NP-DDEs while error analysis, histogram studies and regression metrics further endorse worth of the solvers.

The organizational plan of this study comprises of the following: In the second section, necessary information of LP/NP DDEs is presented. In the third section, a brief description of proposed algorithms and their implementation on five different IVPs having first, second and third order representation on LP/NP-DDEs. In the fourth section, the outcomes of the proposed scheme are summarized in the form of concluding remarks along with provision of future related studies.

II. LINEAR AND NON-LINEAR PANTOGRAPH DELAY DIFFERENTION EQUATIONS

The six different problems based on linear/nonlinear pantograph delay differential equations (LP/NP-DDE) are presented in this section.

Problem 4.1: IVP of LP-DDE of first order for $h(x) = 0.5x$ and $c_1 = 0.1$ in equation (1) as follows [26], [66]–[68]

$$\frac{df}{dx} = 0.5e^{0.5x}f(0.5x) + 0.5f(x), \quad f(0) = 1 \tag{4}$$

The exact solution of equation (4) is given as follows

$$f(x) = e^{x} \tag{5}$$

Problem 4.2: IVP of NP-DDE of first order for $h(x) = 0.5x$ and $c_1 = 0$ in equation (1) as follows [26]

$$\frac{df}{dx} = 1 - 2f^2(0.5x), \quad f(0) = 0, \tag{6}$$

The exact solution of equation (6) is given as follows

$$f(x) = \sin x \tag{7}$$

Problem 4.3: IVP of LP-DDE of first order for $h(x) = 0.5\,x$ and $c_1 = 0$ in equation (1) as follows [8], [26], [67]

$$\frac{df}{dx} = -f(x) + 0.25f(0.5x) - 0.25e^{-0.5x},$$

$$f(0) = 1, \tag{8}$$

The exact solution of equation (8) is given as follows

$$f(x) = e^{-x} \tag{9}$$

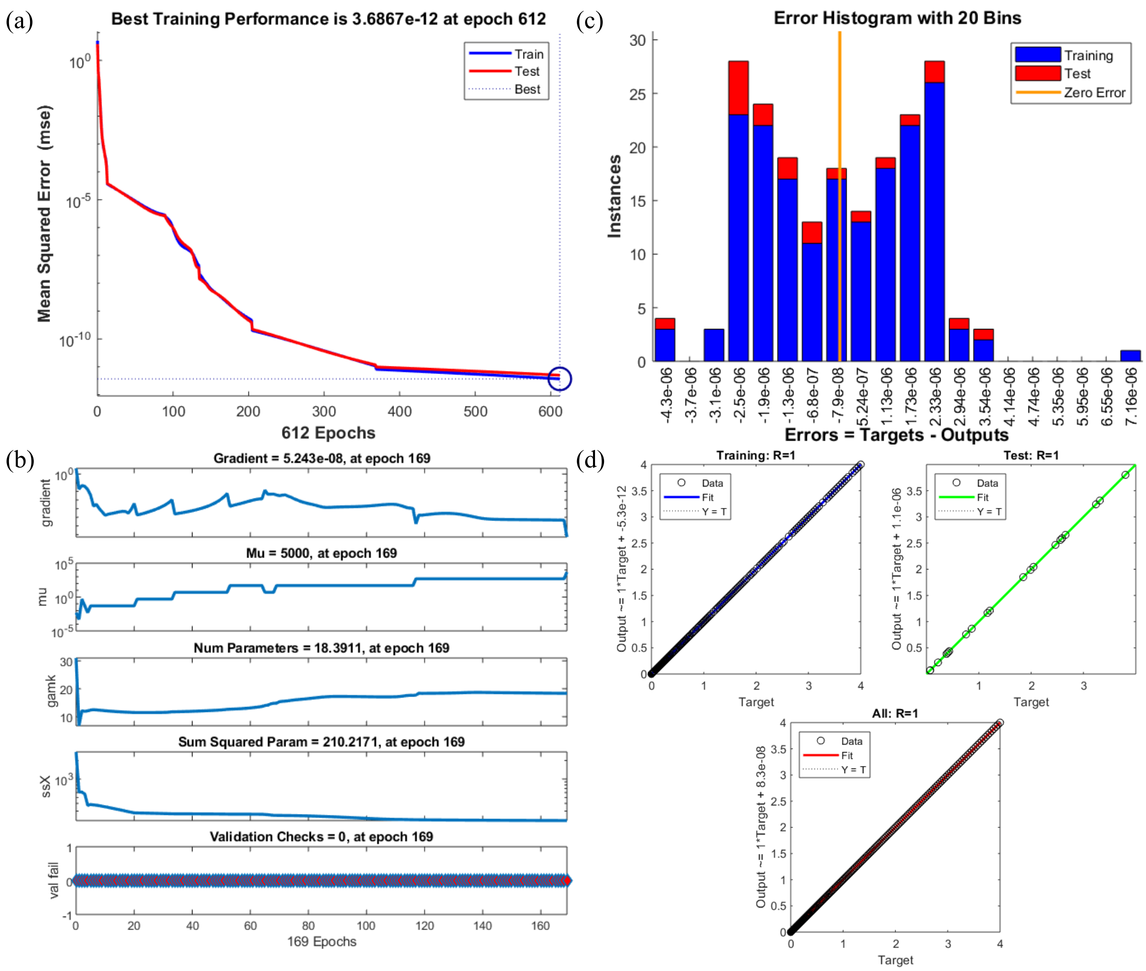

FIGURE 16. Execution of ANN-BRM for solving LP-DDE in problem 4.4 (a) performance plots (b) Training state parameters (c) Error Histogram (d) Regression plots.

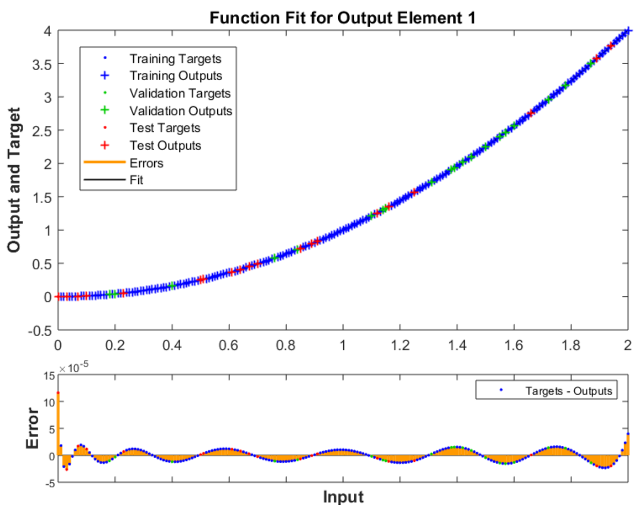

FIGURE 17. Approximate solution with error analysis for ANN-LMM for problem 4.4.

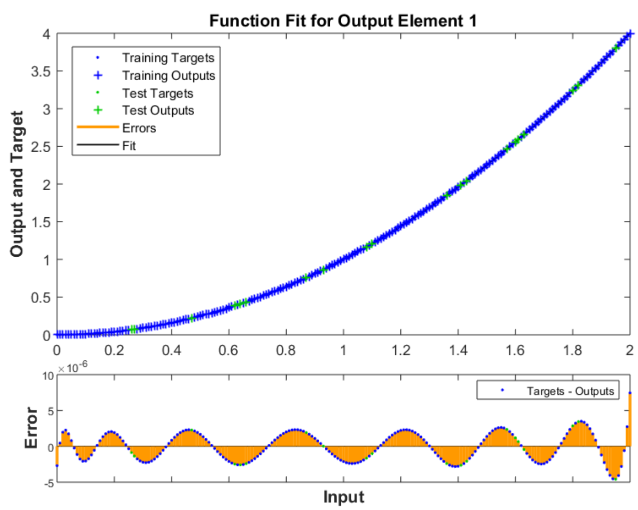

FIGURE 18. Approximate solution with error analysis for ANN-BRM for problem 4.4.

Problem 4.4: IVP of LP-DDE of second order with $h(x) = 0.5x$ and $c_1 = c_2 = 0$ in equation (2) as follows [26], [68]

$$\frac{d^2 f}{dx^2} = 0.75 f(x) + f(0.5x) - x^2 + 2,$$

$$f(0) = 0, \quad \frac{d}{dx} f(0) = 0, \tag{10}$$

The exact solution of equation (10) is given as follows

$$f(x) = x^2 \tag{11}$$

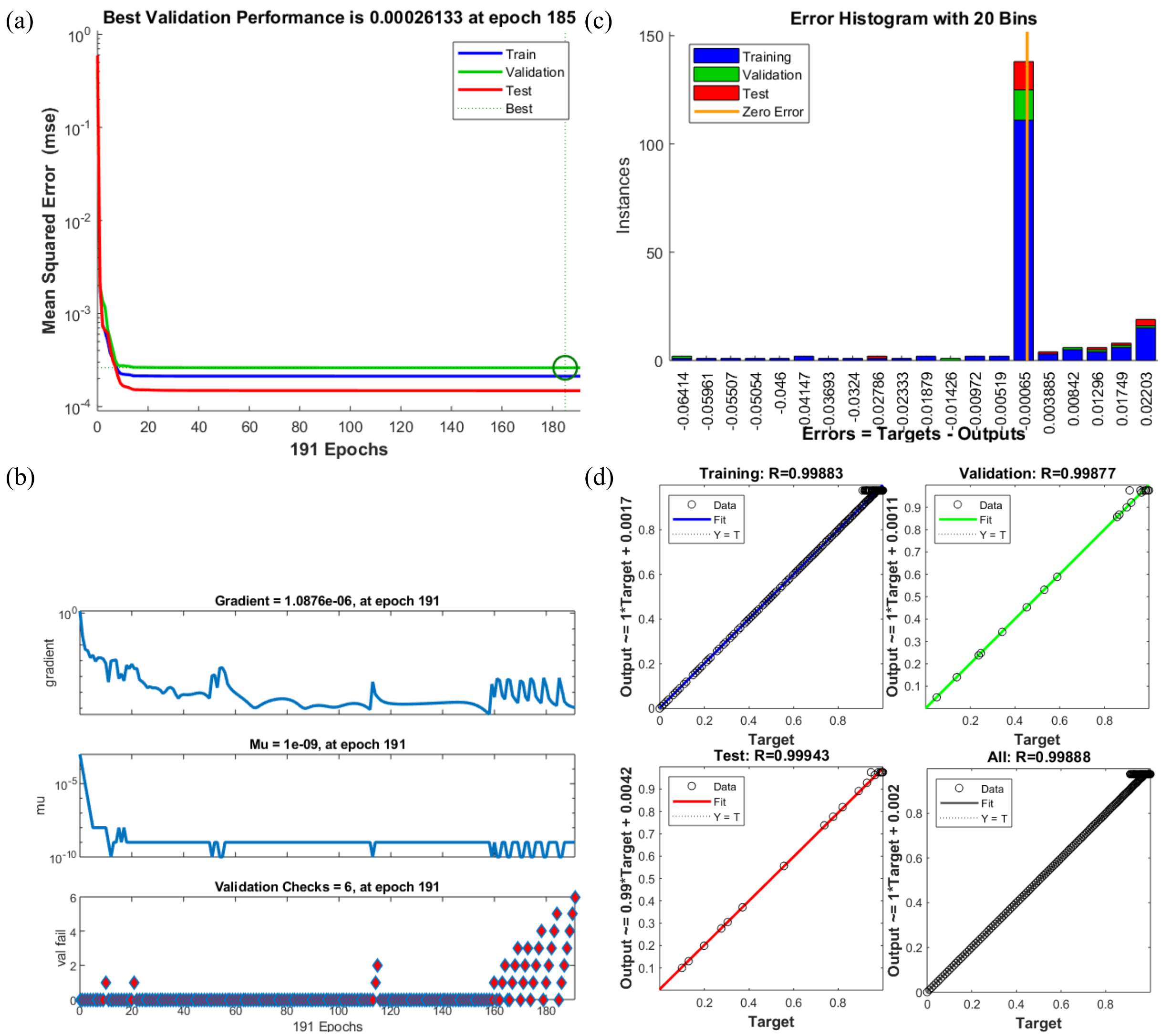

FIGURE 19. Execution of ANN-LMM for solving LP-DDE in problem 4.5 (a) performance plots (b) Training state parameters (c) Error Histogram (d) Regression plots.

Problem 4.5: IVP of LP-DDE of third order with $h(x) = 0.5x$ and $c_1 = c_2 = 0$ in equation (2) as follows [26]

$$\frac{d^3 f}{dx^3} = -1 + 2\,(f(0.5x))^2\,,$$
$$f(0) = 0, \quad \frac{d}{dx} f(0) = 1, \ \frac{d^2}{dx^2} f(0) = 0, \qquad (12)$$

The exact solution of equation (12) is given as follows

$$f(x) = \sin x \qquad (13)$$

III. NUMERICAL EXPERIMENTATION WITH DISCUSSION

The brief description of the methodology adopted and results of the numerical experimentation of the proposed ANN-LMM and ANN-BRM for five different problems. i.e., 4.1-4.5, based on LP/NP-DDEs are presented in this section.

The step wise process flow structure of the proposed methodology is presented in Fig. 1 in five stages, i.e., Problem definition, Mathematical representation, Neural Network Models, Approximate Solutions and Assessment Analysis. ANN-LMM and ANN-BRM are implemented by using 'nftool' routine of neural network toolbox in Matlab for two layers structure (single input, hidden and output) of feedforward networks with backpropagation of LMM and RBM. The architecture of ANNs based on ten number of neurons with log-sigmoid activation function is shown in Fig. 2.

The reference dataset for ANN-LMM and ANN-BRMs for problem 4.1, 4.2, 4.3, 4.4 and 4.5 is generated using equations (8), (10), (12), (14) and (16) for 201 input grids between interval [0, 2]. Now, 70% of data is used arbitrarily for training while 15% is used for both the testing and validation process in case of fitting tool of 2 layered structure of feed forward ANN with backpropagation with LMM and BRM to solve all five problems of LP/NP-DDEs. Training data is utilized for the formulation of the approximate solution on the basis of MSE based merit function, validation data is used for modeling of the neural networks, while testing data is used for assessment of the performance of unbiased inputs.

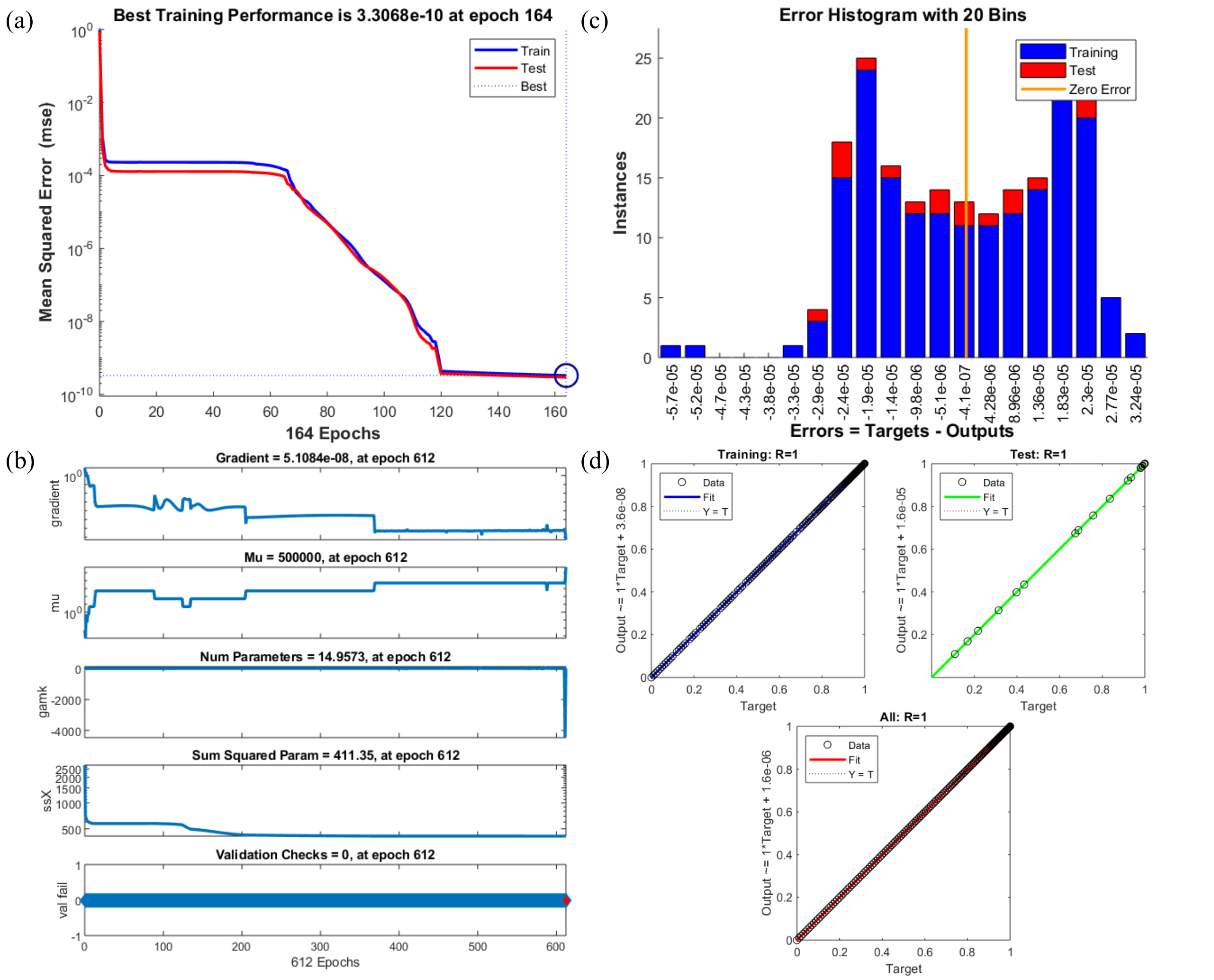

FIGURE 20. **Execution of ANN-BRM for solving NP-DDE in problem 4.5 (a) performance plots (b) Training state parameters (c) Error Histogram (d) Regression plots.**

The performance of ANN-LMM and ANN-BRM in terms of fitness values (i.e., MSE), epochs, performance, back-propagation measures and time for execution are tabulated in Tables 1 and 2, respectively, for all five LP/NP-DDEs in problems 4.1 to 4.5. The values of performance for ANN-LMM are around 10^{-10} to 10^{-04} and 10^{-13} to 10^{-10} for ANN-RBM. MSE values for training, testing and validation are around 10^{-10} to 10^{-04} for ANN-LMM. While the values for training and testing for ANN-RBM lie around 10^{-13} to 10^{-10}. Time complexity in the form of executing time taken by ANN-LMM and ANN-RBM to adjust the weights are given in Table 1 and 2. There is no noticeable difference manifested in computational time of both backpropagation methodologies. Generally, these results show that both approaches give consistently accurate processing, however, ANN-LMM based method is relatively more efficient as compared to ANN-RBM for solving LP/NP-DDEs.

The results of ANN-LMM and ANN-BRM for LP-DDE in problem 4.1 by means of MSE based performance, state transitions parameters, error histograms studies and regression plots are illustration in, Figs 3-4, respectively, while the approximate solutions with error, differences between proposed results and exact solutions, is presented in Fig. 4-5, respectively. Accordingly, the results of both ANN-LMM and ANN-BRM for LP/NP-DDE in problem 4.2, 4.3, 4.4 and 4.5 are given in Figs 7-10, Figs 11-14, Figs 15-18 and Figs 19-22, respectively.

In the subfigures 3(a)-4(a), 7(a)-8(a), 11(a)-12(a), 15(a)-16(a), 19(a)-20(a), the performance of MSE for training, validation and test data against epoch index are shown for LP/NP-DDEs in problems 4.1 to 4.5, respectively. One may see that the best curves of the network are achieved at 1000, 16, 1000, 1000 and 185 epochs with MSE around 10^{-11} to 10^{-10}, 10^{-04}, 10^{-11} to 10^{-10}, 10^{-10} to 10^{-09} and 10^{-04} to 10^{-03} for ANN-LMM, while 608, 169, 1000, 612 and 164 epochs with MSE around 10^{-11} to 10^{-10}, 10^{-11} to 10^{-10}, 10^{-10} to 10^{-09}, 10^{-12} to 10^{-11} and 10^{-10} to 10^{-09} for ANN-RBM in case of problems 4.1 to 4.5, respectively.

The gradient and step size *Mu* of both backpropagation algorithms are presented in the subfigures 3(a)-4(b), 7(b)-8(b), 11(b)-12(b), 15(b)-16(b), 19(b)-20(b), for LP/NP-DDEs in problems 4.1, 4.2, 4.3, 4.4 and 4.5, respectively.

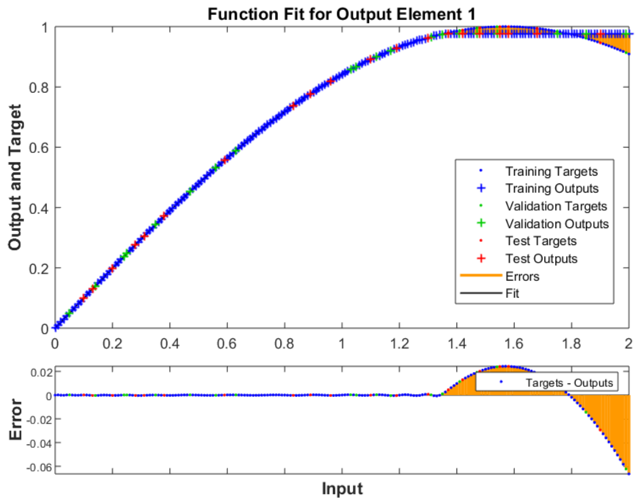

FIGURE 21. **Approximate solution with error analysis for ANN-LMM for problem 4.5.**

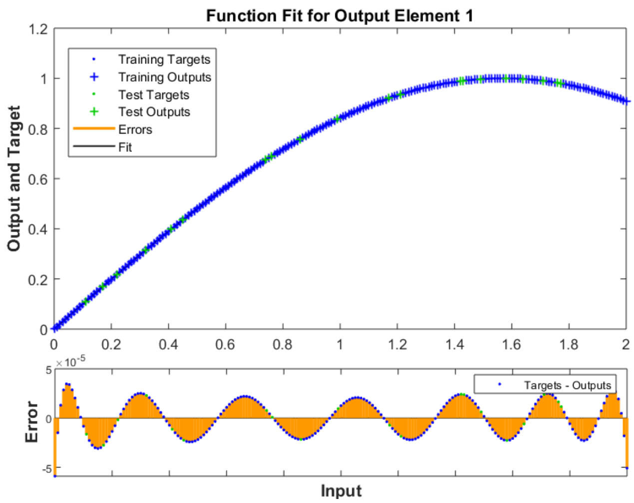

FIGURE 22. **Approximate solution with error analysis for ANN-BRM for problem 4.5.**

The gradient and *Mu* values for LMM based backpropagation are around 10^{-08} to 10^{-04} and 10^{-10} to 10^{-07}, respectively, while, these respective values for RBM are around 10^{-08} to 10^{-04} and 10^{-10} to 10^{-07}. The small variation in the parameters of gradient and *Mu* established the relatively stable performance of RBM over LMM, particularly for NP-DDE in problems 4.2 and 4.5.

The comparison of approximate solutions of ANN-LMM and ANN-RBM with reference exact solutions presented in Figures 5-6, 9-10, 13-14, 17-18 and 21-22, for respective problems 4.1 to 4.5, show the consistent overlapping of both results with 5 to 6 decimal places of accuracy. Moreover, the performance of ANN-LMM for NP-DDE is problem 4.2 and 4.5 is relative inferior from ANN-RBM. The consistent precise performance of ANN-RBM is achieved for both LP/NP-DDE.

The error analysis through histograms are conducted for both ANN-LMM and ANN-RBMs and results are presented in the subfigures 3(c)-4(c), 7(c)-8(c), 11(c)-12(c), 15(c)-16(c), 19(c)-20(c), for LP/NP-DDEs in problems 4.1, 4.2, 4.3, 4.4 and 4.5, respectively. The error bin with reference zero error line of ANN-LMM is found in close vicinity of 10^{-05}, while for ANN-BRM is around 10^{-06} to 10^{-05}. The results evidently show that there is no noticeable difference in the performance for LP-DDE while NP-DDE the small error values are attained by ANN-RBM.

The regression analysis for training, testing, validation and for the whole data is performed for both ANN-LMM and ANN-RBMs and results are presented in the subfigures 3(d)-4(d), 7(d)-8(d), 11(d)-12(d), 15(d)-16(d), 19(d)-20(d), for LP/NP-DDEs in problems 4.1, 4.2, 4.3, 4.4 and 4.5, respectively. It is seen that the value of correlation $R = 1$ for LP-DDE for both ANN-LMM and ANN-RBMs while for NP-DDE the value of $R = 1$ for ANN-RBM while $R = 0.98$ for ANN-LMM close to 1.

IV. CONCLUSION

The strength of two-layered structure of feed-forward artificial neural networks with backpropagation of Levenberg-Marquardt as well as Bayesian Regularization is exploited to find accurate, reliable and stable solutions of initial value problems of linear/nonlinear pantograph delay differential equations. The dataset for training, testing and validation is created from available reported studies of linear/nonlinear pantograph delay differential equations. The both proposed ANN-LMM and ANN-BRMs are implemented on the said dataset for approximate modeling of the system on mean squared error based merit function and learning of weights is conducted with LMM and BRMs. The performance of the designed algorithms ANN-LMM and ANN-BRM on IVPs on first, second and third order LP/NP-DDEs is evaluated and both methodologies are found reasonable precise with matching of order around 5 to 7 decimal places of accuracy, however, the performance of ANN-BRM in all five LP/NP-DDEs is consistent while the result of ANN-LMM is degraded in case of problem 4.2 and 4.5. The absolute error analysis, histogram studies and regression indices also validate the performance of both ANN-LMM and ANN-RBM with relative better performance of ANN-RBM for solving the variants of LP/NP DDEs.

In future, one may look into the strength of ANN-LMM and ANN-RBM with different sigmoidal, radial base and wavelet activation functions to solve neutral, singular, multi, fuzzy, fractional variants of LP/NP DDEs along with their systems. Metaheuristic techniques including GA, PSO, SGA and QPSO can be implemented for training of neural networks for improved performance [69]–[73].

CONFLICTS OF INTEREST

The authors declare that they have no competing interests.

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