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New algorithm for the approximate solution of generalized seventh order Korteweg-Devries equation arising in shallow water waves

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ABSTRACT

In the present article, we present for the first time optimal auxiliary function method (OAFM) for partial differential equation (PDEs). To find efficient and precision the proposed method, we take Lax's seventh order korteweg-de Vries (KdV) and seventh order Sawada Kotera (SK) equations as test examples. The beauty of the planned method lies in auxiliary functions A_i and some parameters C_i which ensure a very rapid convergence of the solution. We compare the approximate solutions got by the proposed method with the homotopy perturbation method (HPM), the Optimal Homotopy asymptotic method. It should be emphasized that very good approximation is obtained at the first iteration. It has been shown, that OAFM is a simple and convergent method for the solution of nonlinear equations. The numerical results rendering that the applied method is explicit, efficacious and facile to utilize, for handling more general nonlinear equations.

Introduction

Most of the physical phenomenon are modeled by nonlinear differential equations.

In such circumstance, it is difficult to get the true solution of these nonlinear differential equations. Lately, numerous authors focused on investigation solitonic equation of nonlinear propblems by utilizing an assortment of incredible techniques, for example, the variational iteration method (VIM) [1,2], homotopy perturbation method (HPM) [3–5], Exp-function method [6–9], sine–cosine method [10], meshless collocation methods [11–17], homogeneous balance method [18–20] and He's frequency formulation and many more [21–32].

In the same manner, we apply a new method, namely called optimal axillary function method (OAFM) for these type different partial differential equations. The beauty of the method is that it gives encouraging results after at only one iteration.

The proposed method was introduced by Marinca et al. in (2018) for finding the approximate analytical solutions for thin film flow of a fourth grade fluid down vertical cylinder and for a Pendulum Wrapping on Two Cylinders [33–35]. Our main work is to extend the proposed method for approximate solutions of Lax's seventh order (KdV) and Seventh order (SK) equations. The Lax's seventh order kdv equation has the following general form [34]

$$\frac{\partial\xi}{\partial t}+35\frac{\partial\xi^4}{\partial\chi}+70\frac{\partial^2\xi^2}{\partial\chi^2}\frac{\partial^3\xi}{\partial\chi^3}+70\frac{\partial\xi}{\partial\chi}\frac{\partial\xi^2}{\partial\chi}+14\frac{\partial\xi}{\partial\chi}\frac{\partial^4\xi}{\partial\chi^4}+21\frac{\partial^3\xi^2}{\partial\chi^2}+28\frac{\partial^2\xi}{\partial\chi^2}\frac{\partial^4\xi}{\partial\chi^4}+\frac{\partial^6\xi}{\partial\chi^6}=0, \tag{1.1}$$

and seventh order SK equation [34] is given as follow

$$\frac{\partial\xi}{\partial t}+63\frac{\partial\xi^4}{\partial\chi}+126\frac{\partial^2\xi^2}{\partial\chi^2}\frac{\partial^3\xi}{\partial\chi^3}+63\frac{\partial\xi}{\partial\chi}\frac{\partial\xi^2}{\partial\chi}+21\frac{\partial\xi}{\partial\chi}\frac{\partial^4\xi}{\partial\chi^4}+21\frac{\partial^3\xi^2}{\partial\chi^2}+21\frac{\partial^2\xi}{\partial\chi^2}\frac{\partial^4\xi}{\partial\chi^4}+\frac{\partial^6\xi}{\partial\chi^6}=0, \tag{1.2}$$

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where $\xi = \xi(\chi, t)$. These equations play an important role in mathematical physics, engineering and applied sciences for investigating travelling solitary wave solutions. Equations (1a) and (1b) are the KdV and Sk equations of the 7th order of Lax, respectively. The names of these two equations are altered because of the coefficient difference of to the region of divergence especially in the case of strongly nonlinear equations.

The whole paper is divided into four sections, the first section contains the introduction, second section is devoted to the fundamental theory of OAFM. In section three the applications of OAFM to different problems are given. In section four the results and discussion of the problems are given while the conclusions of the work are placed in the last section.

Basic idea of OAFM for PDEs

To extent the optimal axillary function method to partial differential equation. We take the general PDE as

$$L[\xi(\chi,t)] = g(\chi,t) + N[\xi(\chi,t)\,] = 0, \tag{2}$$

wwith I.C condition,

$$\Phi\left(\xi, \frac{\partial \xi}{\partial t}\right) \tag{3}$$

Hence L and f show the linear and known functions respectively while N is nonlinear operator.

To obtain the series solution of Eq. (2), we take two terms given as follow,

$$\widetilde{\xi}(\chi,t) = \xi_0(\chi,t) + \xi_1(\chi,t,C_i), \quad i = 1,2,3,\ldots\ldots p \tag{4}$$

For finding initial and first order approximation, we substitute Eqs. (4) into (2). It gives,

$$L[\xi_0(\chi,t)] + L[\xi_1(\chi,t,C_i)] + g(\chi,t) + N[\xi_0(\chi,t) + \xi_1(\chi,t,C_i)] = 0. \tag{5}$$

To find the approximation $\xi_0(\chi,t)$ we take the following equation,

$$L[\xi_0(\chi,t) + g(\chi,t)] = 0, \quad \Phi\left(\xi_0, \frac{\partial \xi_0}{\partial t}\right) = 0, \tag{6}$$

Similarly for first order approximate solution $\xi_1(\chi,t)$,

$$L[\xi_1(\chi,t,C_i)] + N[\xi_0(\chi,t) + \xi_1(\chi,t,C_i)] = 0, \quad \Phi\left(\xi_1, \frac{\partial \xi_1}{\partial t}\right) = 0. \tag{7}$$

Hence, we expend the nonlinear term as follow,

$$N[\xi_0(\chi,t) + \xi_1(\chi,t,C_i)] = N[\xi_0(\chi,t)] + \sum_{k=1}^{\infty} \frac{\xi_1^k}{k!} N^{(k)}[\xi_0(\chi,t)] \tag{8}$$

To avoid the difficulty as we see in solving Eq. (7) and to accelerate the rapid convergence of the first approximation $\xi_1(\chi,t,C_i)$ and implicit of the solution $\widetilde{\xi}(\chi,t)$, instead of the term arising into Eq. (7), we propose another expression, such that Eq. (7) can be written as

$$\begin{gathered} L[\xi_1(\chi,t,C_i)] + \gamma_1[\xi_0(\chi,t)]N[\xi_0(\chi,t)] + \gamma_2\left[\xi_0(\chi,t), C_j\right] = 0, \\ \Phi\left(\xi_1, \frac{\partial \xi_1}{\partial t}\right) = 0. \end{gathered} \tag{9}$$

Remark 1. In Eq. (9) γ_1 and γ_2 are arbitrary auxiliary functions. Which depend on the initial approximation $\xi_0(\chi,t)$ and a number of the unknown parameters C_i and C_j , $i = 1,2,3\ldots, j = s+1, s+2, ..p$.

Remark 2. The auxiliary functions A_1 and A_2 are not unique and are of the same form as $\xi_0(\chi,t)$ is the form of $N[\xi_0(\chi,t)]$ or the combination of both $\xi_0(\chi,t)$ and $N[\xi_0(\chi,t)]$.

Remark 3.

- If $\xi_0(\chi,t)$ or $N[\xi_0(\chi,t)]$ a polynomial functions then the axillary functions should be the sum of sum of polynomial functions.
- If $\xi_0(\chi,t)$ or $N[\xi_0(\chi,t)]$ an exponential functions then then the axillary functions should be the sum of exponential functions.
- If $\xi_0(\chi,t)$ or $N[\xi_0(\chi,t)]$ a trigonometric functions then then the axillary functions should be the sum of trigonometric functions.
- If in the special case $N[\xi_0(\chi,t)] = 0$ then it is clear that $\xi_0(\chi,t)$ is an exact solution of Eq. (2)

Least Square method: for finding the convergence control parameters C_iC_j with help of least square method. For this we introduce the following functional which containing the convergence control parameters in the given domain.

$$\overline{\Delta}(C_i, C_j) = \int_0^t \int_\Omega R^2(\chi, t; C_i, C_j)\, d\chi dt, \tag{10}$$

where R shows the residual,

$$\begin{aligned} R(\chi,t,C_i,C_j) = L\left[\widetilde{\xi}(\chi,t,C_i,C_j)\,\right] + g(\chi,t) + N\left[\widetilde{\xi}(\chi,t,C_i,C_j)\,\right], \quad i \\ = 1,2,..s, \ \ j = S+1, S+2, ..p, \end{aligned}$$

To find the numerical values of convergence control parameters, we differentiate the functional $\overline{\Delta}$ with respect to $C_i$and solving the following system of equation,

$$\frac{\partial \overline{\Delta}_1}{\partial C_1} = \frac{\partial \overline{\Delta}_2}{\partial C_2} = \frac{\partial \overline{\Delta}_3}{\partial C_3} \ldots\ldots \frac{\partial \overline{\Delta}_m}{\partial C_m} = 0 \tag{11}$$

Implementation of OAFM

In this part, to represent the effectiveness and exactness of the proposed technique, we have accomplished series solution of seventh order Lax's Kdv and seventh order Sawada Kotera equations. All computational work has done with help of Mathematica 10.0.

Lax's seventh order KDV equation

Firstly, we take the Lax's seventh order Kdv equation,

$$\begin{aligned} &\frac{\partial \xi}{\partial t} + 35\frac{\partial \xi^4}{\partial \chi} + 70\frac{\partial^2 \xi^2}{\partial \chi^2}\frac{\partial^3 \xi}{\partial \chi^3} + 70\frac{\partial \xi}{\partial \chi}\frac{\partial \xi^2}{\partial \chi} + 14\frac{\partial \xi}{\partial \chi}\frac{\partial^4 \xi}{\partial \chi^4} + 21\frac{\partial^3 \xi^2}{\partial \chi^2} + 28\frac{\partial^2 \xi}{\partial \chi^2}\frac{\partial^4 \xi}{\partial \chi^4} + \frac{\partial^6 \xi}{\partial \chi^6} \\ &= 0, \end{aligned} \tag{12}$$

with I.C

$$\xi(\chi,0) = 2\alpha^2\left(\mathrm{sech}^2(\alpha\chi)\,\right) \tag{13}$$

Here α is an arbitrary constant and the exact solution has been found for Eq. (12) in [34],

$$\xi(\chi,t) = 2\alpha^2\left(\mathrm{sech}^2\left(\alpha\left(\chi - 64\alpha^6 t\right)\,\right)\,\right) \tag{14}$$

Linear and nonlinear terms in Eq. (12) are given as follow,

$$L(\xi,t)=\frac{\partial\xi(\chi,t)}{\partial t}. \tag{15}$$

$$g(\chi,t)=0. \tag{16}$$

$$N(\xi)=35\frac{\partial\xi^4}{\partial\chi}+70\frac{\partial^2\xi^2}{\partial\chi^2}\frac{\partial^3\xi}{\partial\chi^3}+70\frac{\partial\xi}{\partial\chi}\frac{\partial\xi^2}{\partial\chi}+14\frac{\partial\xi}{\partial\chi}\frac{\partial^4\xi}{\partial\chi^4}+21\frac{\partial^3\xi^2}{\partial\chi^2}+28\frac{\partial^2\xi}{\partial\chi^2}\frac{\partial^5\xi}{\partial\chi^5}+\frac{\partial^6\xi}{\partial\chi^6}. \tag{17}$$

Using Eq. (6), we get the initial value $\xi_0(\chi,t)$,

$$\frac{\partial\xi_0(\chi,t)}{\partial t}=0,\quad \xi_0(\chi,0)=2\alpha^2\left(\text{sech}^2(\alpha\chi)\right) \tag{18}$$

The solution of Eq. (18) is

$$\zeta_0(\eta,t)=2\alpha^2\left(\text{sech}^2(\alpha\eta)\right) \tag{19}$$

By substituting Eq. (19) into Eq. (17), the nonlinear operator becomes

$$\begin{aligned}N[\xi_0(\chi,t)]=35\frac{\partial\xi_0^4}{\partial\chi}+70\frac{\partial^2\xi_0^2}{\partial\chi^2}\frac{\partial^3\xi_0}{\partial\chi^3}+70\frac{\partial\xi_0}{\partial\chi}\frac{\partial\xi_0^2}{\partial\chi}+14\frac{\partial\xi_0}{\partial\chi}\frac{\partial^4\xi_0}{\partial\chi^4}\\+21\frac{\partial^3\xi_0^2}{\partial\chi^2}+28\frac{\partial^2\xi_0}{\partial\chi^2}\frac{\partial^4\xi_0}{\partial\chi^4}+\frac{\partial^6\xi_0}{\partial\chi^6}.\end{aligned} \tag{20}$$

The first approximation $\xi_1(\chi,t)$ is given by Eq. (9)

$$\begin{aligned}\frac{\partial\xi_1(\chi,t)}{\partial t}+\gamma_1[\xi_0(\chi,t),C_i]N[\xi_0(\chi,t)]+\gamma_2\left[\xi_0(\chi,t),C_j\right]=0,\\ \xi_1(\chi,0)=0,\end{aligned} \tag{21}$$

Using the OAFM procedure we choose the axillary functions as follow,

$$\begin{cases}\gamma_1=C_1[\text{sech}(\chi)]^2+C_2[\text{sech}(\chi)]^4.\\ \gamma_2=C_3[\text{sech}(\chi)]^6+C_4[\text{sech}(\chi)]^8.\end{cases} \tag{22}$$

Using Eqs. (19), (22) into Eq. (21), we get the first approximation as

$$\xi_1(\chi,t)=\left(\begin{array}{c}t\text{sech}^4(\chi)((-\text{sech}^2(\chi)(C_3+C_4\text{sech}^2(\chi)-64\alpha^8(C_1+2C_2+C_1cosh(2\chi))\\ \text{sech}^2(\chi\alpha)-224\alpha^8(-9+16\alpha^2)(C_1+2C_2+C_1cosh(2\chi))\text{sech}^4(\chi\alpha)+1792\alpha^8\\ (-5+18\alpha^2)(C_1+2C_2+C_1cosh(2\chi))\text{sech}^6(\chi\alpha)-560\alpha^8(-13+120\alpha^2)\\ (C_1+2C_2+C_1cosh(2\chi))\text{sech}^8(\chi\alpha)+40320\alpha^{10}(C_1+2C_2+C_1cosh(2\chi))\\ \text{sech}^{10}(\chi\alpha)+448\alpha^7(3+\alpha^2)\text{sech}^5(x\alpha)(C_1sinh(x(-2+\alpha))+2(C_1+2C_2)\\ sinh(\chi\alpha)+C_1sinh(\chi(2+\alpha))+280\alpha^7(-9-12\alpha^2+64\alpha^4)\text{sech}^7(\chi\alpha)\\ (C_1sinh(\chi(-2+\alpha))+2(C_1+2C_2)sinh(\chi\alpha)+C_1sinh(\chi(2+\alpha))-4480\alpha^9\\ (-1+17\alpha^2)\text{sech}^9(\chi\alpha)(C_1sinh(\chi(-2+\alpha))+2(C_1+2C_2)sinh(\chi\alpha)+C_1\\ sinh(\chi(2+\alpha))+67200\alpha^{11}\text{sech}^{11}(\chi\alpha)(C_1sinh(\chi(-2+\alpha))+2(C_1+2C_2)\\ sinh(\chi\alpha)+C_1sinh(\chi(2+\alpha))\end{array}\right) \tag{23}$$

Adding Eqs. (19) and (23), we obtain the series solution in the following expression,

$$\widetilde{\xi}(\chi,t)=\xi_0(\chi,t)+\xi_1(\chi,t,C_1,C_2,C_3,C_4). \tag{24}$$

Seventh order SK equation

The seventh order Sawada Kotera equation is given as follow,

$$\frac{\partial\xi}{\partial t}+63\frac{\partial\xi^4}{\partial\chi}+126\frac{\partial^2\xi^2}{\partial\chi^2}\frac{\partial^3\xi}{\partial\chi^3}+63\frac{\partial\xi}{\partial\chi}\frac{\partial\xi^2}{\partial\chi}+21\frac{\partial\xi}{\partial\chi}\frac{\partial^4\xi}{\partial\chi^4}+21\frac{\partial^3\xi^2}{\partial\chi^2}+21\frac{\partial^2\xi}{\partial\chi^2}\frac{\partial^4\xi}{\partial\chi^4}+\frac{\partial^6\xi}{\partial\chi^6}=0, \tag{23}$$

with I.C

$$\xi(\chi,0)=\frac{4}{3}k^2(2-3(tanh(\alpha\chi))) \tag{24}$$

Here α, k are arbitrary constants. The exact solution for Eq. (23) can be found in [34]

$$\xi(\chi,t)=\frac{4}{3}k^2\left(2-3tanh^2\left(\alpha\left(\chi-\frac{256\alpha^6t}{3}\right)\right)\right) \tag{25}$$

Using the same procedure like above problem, we have the initial approximate $\xi_0(\chi,t)$ is given,

$$\frac{\partial\xi_0(\chi,t)}{\partial t}=0,\quad \xi_0(\chi,0)=\frac{4}{3}k^2(2-3(tanh(\alpha\chi))), \tag{26}$$

We get the solution for Eq. (26) is,

$$\xi_0(\chi,t)=\frac{4}{3}k^2(2-3(tanh(\alpha\chi))) \tag{27}$$

By substituting Eq. (27) into Eq. (23), the nonlinear operator becomes

$$\begin{aligned}N[\xi_0(\chi,t)]=63\frac{\partial\xi_0^4}{\partial\chi}+126\frac{\partial^2\xi_0^2}{\partial\chi^2}\frac{\partial^3\xi_0}{\partial\chi^3}+63\frac{\partial\xi_0}{\partial\chi}\frac{\partial\xi_0^2}{\partial\chi}+21\frac{\partial\xi_0}{\partial\chi}\frac{\partial^4\xi_0}{\partial\chi^4}+21\frac{\partial^3\xi_0^2}{\partial\chi^2}\\+21\frac{\partial^2\xi_0}{\partial\chi^2}\frac{\partial^4\xi_0}{\partial\chi^4}+\frac{\partial^6\xi_0}{\partial\chi^6}.\end{aligned} \tag{28}$$

The first approximation $\xi_1(\chi,t)$ is given by Eq. (9)

$$\begin{aligned}\frac{\partial\xi_1(\chi,t)}{\partial t}+\gamma_1[\xi_0(\chi,t),C_i]N[\xi_0(\chi,t)]+\gamma_2\left[\xi_0(\chi,t),C_j\right]=0,\\ \xi_1(\chi,0)=0.\end{aligned} \tag{29}$$

Here γ_1,γ_2 are chosen according to initial approximation,

$$\begin{cases}\gamma_1=C_1(\text{Tanh}(\chi))^2+C_2(\text{Tanh}(\chi))^4.\\ \gamma_2=C_3(\text{Tanh}(\chi))^6+C_4(\text{Tanh}(\chi))^8.\end{cases} \tag{30}$$

Using Eq. (27), (28) and (30) into Eq. (29), we get the first approximation as

$$\xi_1(\chi,t;C_i)=\begin{pmatrix} -\frac{1}{3}ttanh^2(\chi)\left(\alpha^7(C_1-C_2+(C_1+C_2)cos(2\chi)\right)\text{sech}^2(2\chi)\text{sech}^{10}(\chi\alpha)(147\alpha \\ \left(-25+192\alpha^2\right)-42\alpha(11+100\alpha^2)cosh(2\chi\alpha)+2856\alpha cosh(4\chi\alpha)-6\alpha \\ (59+336\alpha^2)cosh(2\chi\alpha)+3\alpha cosh(8\chi\alpha)+98(93+80\alpha^2)sinh(2\chi\alpha)+14 \\ (201-664\alpha^2)sinh(4\chi\alpha)-42(27+16\alpha^2)sinh(6\chi\alpha)+7(3+8\alpha^2)sinh(8\chi\alpha) \\ +3tanh^4(\chi)(C_3+C_4tanh^2(\chi). \end{pmatrix} \tag{31}$$

Adding Eqs. (27) and (31), we acquire the first order series solution by the following expression,

$$\widetilde{\xi}(\chi,t)=\xi_0(\chi,t)+\xi_1(\chi,t,C_1,C_2,C_3,C_4). \tag{32}$$

Numerical results

In this section, we illustrate the accuracy of our procedure for an arbitrary constant $\alpha=0.1$, also we show the comparison of absolute errors with Homotopy perturbation method (HPM) and Optimal Homotopy asymptotic method (OHAM) for different values of the time.

4.1: For finding the convergence control parameters C_i, $i=1,2,3..$ we used the least square method. Whose values are given as following.

$$\begin{aligned} C_1&=0.4921659704908877.\\ C_2&=-0.2606759872371297.\\ C_3&=-0.05781845184781508.\\ C_4&=-0.10563165777129231.\end{aligned} \tag{33}$$

By using the above values of C_1, C_2, C_3, C_4 in Eq. (20) we achieve the series solution for the Lax's seventh order Kdv equation.

4.2: Similarly with help of least square method we found the values C_i, $i=1,2,3..$ which are given below, Eq. (35), are

$$\begin{aligned} C_1&=-0.03285281870901489.\\ C_2&=0.0243874058162013.\\ C_3&=1.5782393304235135\times10^{-8}.\\ C_4&=-1.7257248359166504\times10^{-8}.\end{aligned} \tag{37}$$

Using these constants these constants in Eq. (34), we get the first order approximate solution for seventh order SK equation.

To verify the accuracy of the approximate solution if we compare these analytical results with other analytical methods used in literature for these problems. In Tables 1–3 it can be seen that our proposed method gives more accurate results than HPM and OHAM. Figs. 1–3 show the approximate solution, exact solution and absolute errors obtained by OAFM for Eq. (16) while Fig. 4 shows the 2D graph for the approximate solution at different values of t (see Figs. 5 and 6).

Conclusion

In the present work, we extended the new algorithm namely called optimal axillary function method and successfully applied for the approximate solution of Lax's seventh order Kdv and Sawadara Kotera equations. The numerical results obtained by the planned method are compared with those obtained by HPM and OHAM presented in the

Table 1
Comparison of absolute errors of OAFM with HPM, OHAM when $t=0.1$ for Lax's seventh order Kdv equation.

χ	Absolute error HPM [34]	Absolute error OHAM [34]	Absolute error OAFM
0.1	1.523567×10^{-4}	7.09824×10^{-8}	8.88283×10^{-9}
0.2	3.046766×10^{-4}	1.19166×10^{-8}	7.54594×10^{-9}
0.3	4.569652×10^{-4}	7.13055×10^{-9}	$6.07229\times10^{-9}\times10^{-9}$
0.4	6.092284×10^{-4}	1.42267×10^{-8}	4.63123×10^{-9}
0.5	7.614719×10^{-4}	2.12695×10^{-8}	3.36451×10^{-9}

Table 2
Comparison of absolute errors of OAFM with HPM, OHAM when $t=0.3$ for Lax's seventh order Kdv equation.

χ	Absolute error HPM [34]	Absolute error OHAM [34]	Absolute error OAFM
0.1	1.51770×10^{-4}	1.49674×10^{-8}	2.66484×10^{-8}
0.2	3.03446×10^{-4}	5.96959×10^{-8}	2.26378×10^{-8}
0.3	4.55033×10^{-4}	2.68558×10^{-8}	1.82168×10^{-8}
0.4	6.06538×10^{-4}	4.75981×10^{-8}	1.38937×10^{-8}
0.5	7.57965×10^{-4}	6.81050×10^{-8}	1.00935×10^{-8}

Table 3
Comparison of absolute errors of OAFM with HPM, OHAM when $t=0.5$ for Lax's seventh order Kdv equation.

χ	Absolute error HPM [34]	Absolute error OHAM [34]	Absolute error OAFM
0.1	1.50294×10^{-4}	1.37177×10^{-8}	4.4414×10^{-8}
0.2	3.00437×10^{-4}	2.12036×10^{-8}	3.77295×10^{-8}
0.3	4.50436×10^{-4}	5.59122×10^{-8}	3.03613×10^{-8}
0.4	6.00296×10^{-4}	9.02518×10^{-8}	2.31560×10^{-8}
0.5	7.50021×10^{-4}	1.24071×10^{-7}	1.68224×10^{-8}

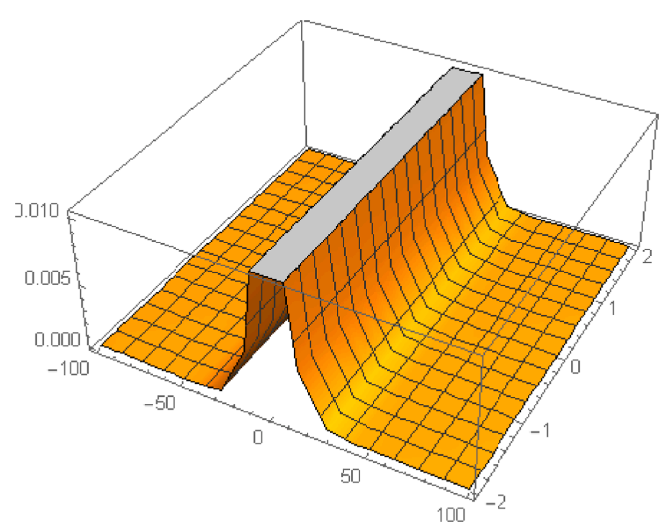

Fig. 1. 3D surface obtained by OAFM solution for Lax's Seventh order Kdv equation.

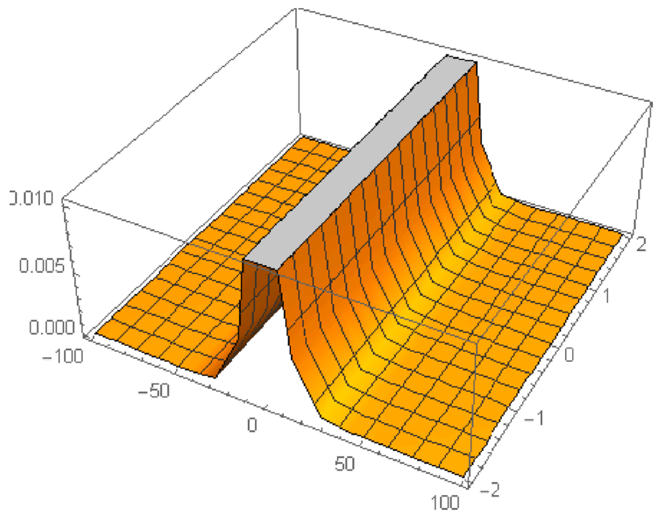

Fig. 2. 3D surface obtained by the exact solution for Lax's Seventh order Kdv equation.

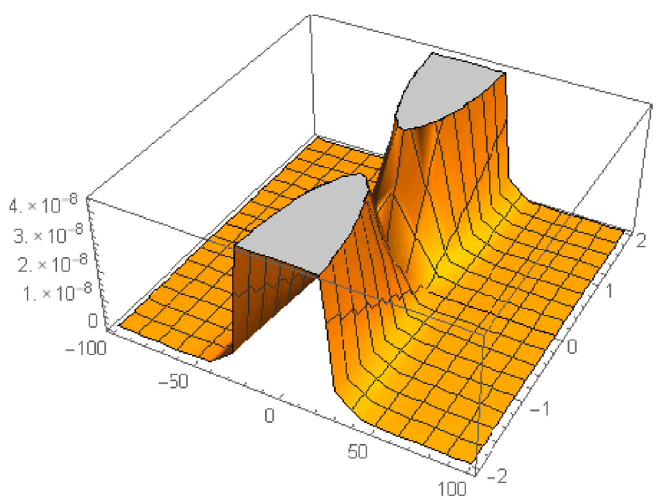

Fig. 3. 3D surface for the absolute errors obtained by OAFM for Lax's Seventh order Kdv equation.

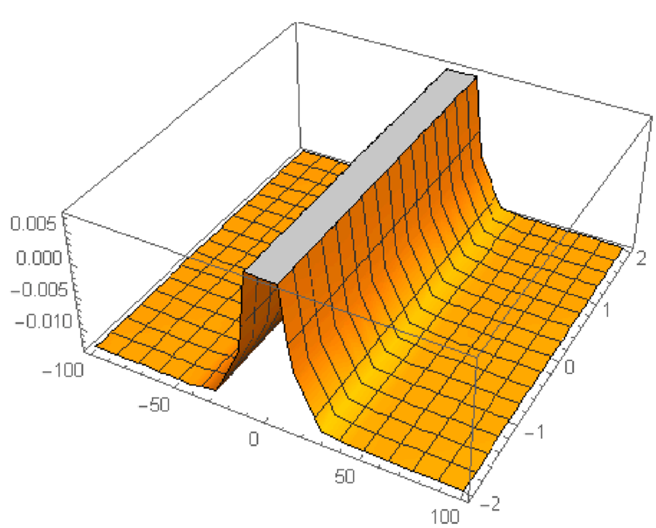

Fig. 4. 3D surface obtained by OAFM solution for Seventh order SK equation.

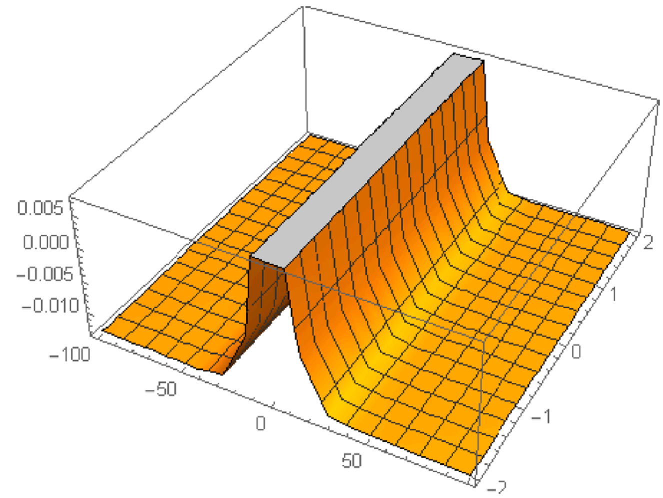

Fig. 5. 3D surface obtained by the exact solution for Seventh order SK equation.

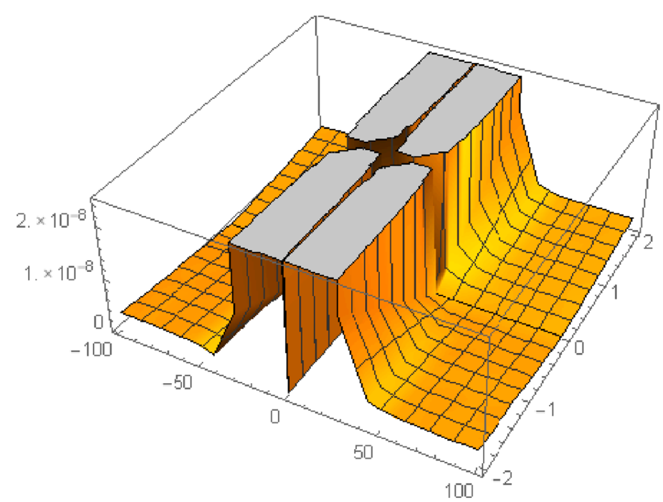

Fig. 6. 3D surface for the absolute errors obtained by OAFM for Seventh order SK equation.

Table 4
Comparison of absolute errors of OAFM with HPM, OHAM when $t = 0.1$ Seventh order SK equation.

η	Absolute error HPM [34]	Absolute error OHAM [34]	Absolute error OAFM
0.1	9.68087×10^{-5}	3.24071×10^{-9}	5.8155×10^{-10}
0.2	1.93593×10^{-4}	1.26255×10^{-9}	8.94501×10^{-10}
0.3	2.90358×10^{-4}	5.77130×10^{-9}	6.62134×10^{-9}
0.4	3.87106×10^{-4}	1.02658×10^{-8}	1.76305×10^{-8}
0.5	4.83840×10^{-4}	1.47269×10^{-8}	3.3821×10^{-8}

literature. Our proposed method is valid if even the nonlinear equation does not contain small or large parameters. The proposed method especially contains A_1 and A_2 auxiliary functions and some parameters $C_1, C_2, \ldots$ which ensure a very rapid convergence of the solution. The proposed method can be applied for different fractional order partial differential equation (FPDEs) and Integro-differential equations (see Tables 4–6).

Table 5
Comparison of absolute errors of OAFM with HPM, OHAM when $t = 0.2$ Seventh order SK equation.

η	Absolute error HPM [34]	Absolute error OHAM [34]	Absolute error OAFM
0.1	9.63425×10^{-5}	1.65768×10^{-8}	2.66484×10^{-9}
0.2	1.92540×10^{-4}	3.16429×10^{-8}	2.26378×10^{-9}
0.3	2.88597×10^{-4}	4.64205×10^{-8}	1.82168×10^{-8}
0.4	3.84516×10^{-4}	6.08325×10^{-8}	1.38937×10^{-8}
0.5	4.80300×10^{-4}	7.48078×10^{-8}	1.00935×10^{-7}

Table 6
Comparison of absolute errors of OAFM with HPM, OHAM when $t = 0.5$ for Seventh order SK equation.

η	Absolute error HPM [34]	Absolute error OHAM [34]	Absolute error OAFM
0.1	9.51987×10^{-5}	7.14596×10^{-8}	2.90717×10^{-9}
0.2	1.90135×10^{-4}	9.91635×10^{-8}	4.47308×10^{-9}
0.3	2.84813×10^{-4}	1.25879×10^{-8}	3.31073×10^{-8}
0.4	3.79236×10^{-4}	1.51446×10^{-8}	8.81531×10^{-8}
0.5	4.73405×10^{-4}	1.757225×10^{-7}	1.68411×10^{-7}

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Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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