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Existence theory and numerical simulations of variable order model of infectious disease

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ABSTRACT

This study uses variable order differentiation and integration to investigate the disease dynamical model of COVID-19. Here, we update the results of the qualitative and quantitative analysis. We obtain necessary conclusions for the existence theory of the solution to the suggested model in order to satisfy the aforementioned criteria using fixed point theories of Banach and Schauder. Additionally, we simulate the outcomes mathematically and graphically using the Euler modified technique for numerical purposes. There are several graphs provided that relate to various variable ordering. In addition, we compare our simulated results with the real data results also in case of infected class.

1. Introduction

Mathematical models are powerful tools to investigate various real world problems. The concept was build in 1927 by McKendrick in the form of simple SIR model known as susceptible, infected and recovered classes. Later on the aforesaid concept was extended to model various complex dynamics of real world problems. Here, for the readers, we refer some famous work as [1–4]. Currently, COVID-19 is a big medical issue throughout the world. The concerned disease was spread out in China at the end of 2019. Within few month the disease took the form of outbreak and this was announced by WHO in April 2020. Approximately, 682.775628 millions people have got the infection up to date. In addition, nearly 6.821478 millions people have been died due to the said infection. Also, 655.687341 people have gotten ride from the infection. Throughout the world the infection has destroyed the health systems of many countries with weak economy (see [5]). Similarly, economical situations all over the world have been suffered very well. Recently, health departments of some countries like USA, China, UK, Germany, etc have been succeed to prepare vaccine for the aforementioned disease. The said vaccines are now available in many countries of the world. Here, for some details we refer [6]. Here it should be kept in minds that the area devoted to mathematical epidemiology in recent times a very hot area of research. In the said area different mathematical models have been studied under the various concepts of classical, fractional and stochastic calculus. Researchers have used the mentioned tools to develop various mathematical models for different infectious diseases and have created significant sound applicable results. For instance, for Hepatitis B and C, Cancer disease, HIV/AIDS, typhoid and

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dengue fever, etc. Recently, the COVID-19 has been modeled very well by using different concepts of fractional calculus, we refer [7–10].

To help the health departments, and physicians, researchers of physical sciences are also trying to investigate the transmission dynamics and predict the future planing to control such diseases. In this regards, large numbers of mathematical models have been prepared. Some famous studies, we refer as [11–13]. Some authors [14] have used SI type model to study the transmission of COVID-19 disease in Portugal for 21 days as

$$\begin{cases} \dot{\mathcal{S}} = -(1-\varrho)k\mathcal{S}(t)\mathcal{I}(t) - k\varrho\omega\mathcal{S}(t)\mathcal{I}(t), \\ \dot{\mathcal{I}} = (1-\varrho)k\mathcal{S}(t)\mathcal{I}(t) + \varrho k\omega\mathcal{S}(t)\mathcal{I}(t) - \frac{1}{\xi}\mathcal{I}(t), \\ \mathcal{S}(t)|_{t=0} = \mathcal{S}_0, \quad \mathcal{I}(t)|_{t=0} = \mathcal{I}_0, \end{cases} \tag{1}$$

where $\mathcal{S}(t)$ is susceptible people, $\mathcal{I}(t)$ is infected people, k is the rate constant, ϱ is rate of isolation and ω is rate of protection. Authors [15] extended the model (1) by involving natural death rate, recruitment rate and taking derivative in fractional order as

$$\begin{cases} \mathrm{D}_t^{\theta}(\mathcal{S}(t)) = \lambda - (1-\varrho)k\mathcal{S}(t)\mathcal{I}(t) - \varrho k\omega\mathcal{S}(t)\mathcal{I}(t) - \mu\mathcal{S}, \ t \in [0,T], \\ \mathrm{D}_t^{\theta}(\mathcal{I}(t)) = (1-\varrho)k\mathcal{S}(t)\mathcal{I}(t) + \varrho k\omega\mathcal{S}(t)\mathcal{I}(t) - \frac{1}{\xi}\mathcal{I}(t) - \mu\mathcal{I}, \ t \in [0,T] \\ \mathcal{S}(t)|_{t=0} = \mathcal{S}_0, \quad \mathcal{I}(t)|_{t=0} = \mathcal{I}_0, \end{cases} \tag{2}$$

where $0 < \theta \leq 1$. They established comprehensive results regarding the existence theory and numerical analysis subject to the usual fractional order derivative .

Here, it is authenticating that variable order calculus is the natural extension of classical calculus. The aforementioned area has been built up by Smko and his co-authors in 1993 whose detail can be found in [16]. Further, some authors applied variable order problems in photoelasticity (see [17]). The stability and convergence of a new explicit finite-difference approach for the variable-order nonlinear fractional diffusion equation was also studied by the authors (see [18]). In the same, way researchers developed numerical schemes using different methods for variable order problems. Here, we refer few works as [19,20], and [21]. In addition, researchers [22], and [23] have established existence theory for variable order problems. Since variable order operators are the natural extension of classical ordinary as well as fractional orders. Therefore, using such operators will provide as sophisticated tools to study the dynamical systems of infectious disease.

Motivated from the aforementioned importance and applications of variable order differentiations and integrations, we consider the model (2) under the variable order as

$$\begin{cases} \mathrm{D}_t^{\theta(x)}(\mathcal{S}(t)) = \lambda - k(1-\varrho)\mathcal{S}(t)\mathcal{I}(t) - \varrho k\omega\mathcal{S}(t)\mathcal{I}(t) - \mu\mathcal{S}, \ x, t \in [0,T], \\ \mathrm{D}_t^{\theta(x)}(\mathcal{I}(t)) = k(1-\varrho)\mathcal{S}(t)\mathcal{I}(t) + \varrho k\omega\mathcal{S}(t)\mathcal{I}(t) - \frac{1}{\xi}\mathcal{I}(t) - \mu\mathcal{I}, \ x, t \in [0,T] \\ \mathcal{S}(t)|_{t=0} = \mathcal{S}_0, \quad \mathcal{I}(t)|_{t=0} = \mathcal{I}_0, \end{cases} \tag{3}$$

where $\theta : [0,T] \to (0,1]$ is continues function in $x \in [0,T]$. We begin by applying the Banach and Schauder fixed point theorems to develop the existence theory for the model under consideration. Numerous works dealing with classic fractional order problems for the existence theory have used the relevant fixed point solutions. We refer few papers for readers as [24–26], and [27]. Further, for numerical interpretation, we use the modified Euler method already studied for various classical and fractional problems in [28–30], and [31,32]. For numerical illustrations, various powerful numerical methods have been introduced like [33–35].

Here we describe that our proposed model is neither classical order nor traditional fractional. But here, we have taken the corresponding derivative in terms of variable order continues function. The variable order differentiations and integrations provide a natural extension of the mentioned operators. There are many physical problems which cannot capture by using classical or fractional order operators. Moreover, important classes of physical phenomena where the order itself is a function of either dependent or independent variables have the ability to clarify the said. Hence, there exist classes of physical problems that would be better described by variable-order operators. We here extend our model under the concept of variable order. In additions, to testify weather our proposed model has a solution or not. This criteria can be verified by using the existence theory of solution for which the classical fixed point theorems play significant roles. Also, numerical investigations is an important aspect of nonlinear analysis, therefore, we will use a powerful numerical method to simulate our theoretical findings. The corresponding results will be graphically presented using Matlab 16.

2. Preliminaries

Here, we recall some definition from [16–18].

Definition 2.1. Let $\theta : [0,T] \to (0,1]$ be continues function, then variable order integration for $f \in L[0,T]$ is defined as

$$I_t^{\theta(x)} f(t) = \frac{1}{\Gamma(\theta(x))} \int_0^t (t-\varsigma)^{\theta(x)-1} f(\varsigma) d\varsigma, \ x, t \in [0,T],$$

such that the right side converges point wise.

Definition 2.2. For the continuous function $\theta : [0, T] \to (0, 1]$, variable order derivative for $f \in C[0, T]$ is defined by

$$D_t^{\theta(x)} f(t) = \frac{1}{\Gamma(1-\theta(x))} \int_0^t (t-\varsigma)^{-\theta(x)} f'(\varsigma) d\varsigma, \ x, t \in [0, T],$$

provided that integral on right side converges.

Lemma 2.3 ([18]). *If $\theta : [0, T] \to (0, 1]$, and $f \in C[0, T] \cup L(0, T)$, then the solution of variable order problem with $g \in L[0, T]$*

$$D_t^{\theta(x)} f(t) = h(t), \ x, t \in [0, T],$$

is described as

$$f(t) = a_0 + \frac{1}{\Gamma(\theta(x))} \int_0^t (t-\varsigma)^{\theta(x)-1} f(\varsigma) d\varsigma, \ x, t \in [0, T].$$

Here, we derive some data dependence results for the given model (3).

Lemma 2.4. *The feasible region for the proposed model solution is described as*

$$\mathbf{\Omega} = \left\{ (\mathcal{S}, \mathcal{I}) \in \mathbf{R}_+^2 : \ 0 < \mathcal{N}(t) \leq \frac{\lambda}{\mu} \right\}.$$

Proof. Let $\mathcal{N}$ be the total population of the community with $\mathcal{N}(0) = \mathcal{N}_0$ be initial value, then one has

$$\mathcal{N}(t) = \mathcal{S}(t) + \mathcal{I}(t). \tag{4}$$

Taking derivative of order $\theta(x)$ of (4), we have

$$\begin{aligned} D_t^{\theta(x)} \mathcal{N}(t) &= D_t^{\theta(x)} \mathcal{S}(t) + D_t^{\theta(x)} \mathcal{I}(t) \\ &\leq \lambda - \mu(\mathcal{S}(t) + \mathcal{I}(t)) \\ &= \lambda - \mu \mathcal{N}(t). \end{aligned} \tag{5}$$

Using Laplace transform corresponding to variable t of (5), one has

$$\mathcal{N}(t) \leq \left[\frac{\lambda + \mu \mathcal{N}_0}{\mu} \right] t^{\theta(x)} E_{\theta(x)} \left(-\mu t^{\theta(x)} \right) + \frac{\lambda}{\mu}, \tag{6}$$

as $t \to \infty$, (6) yields that $\mathcal{N}(t) \leq \frac{\lambda}{\mu}$, hence the required result is received. □

Further, the disease free equilibrium point for (3) can be calculated as done in [15] as $\mathcal{E}_0 = \left(\frac{\lambda}{\mu}, \ 0 \right)$, and in the same way the unique pandemic equilibrium point is given by

$$\mathcal{E}^* = \left(\frac{1 + \xi\delta}{\xi(k(1-\varrho) + \varrho k \omega)}, \ \frac{a\xi(k(1-\varrho) + \varrho k\omega) - \delta(1+\xi\delta)}{(k(1-\varrho) + \varrho k\omega)(1+\xi\delta)} \right).$$

In addition, the basic reproduction number has been given in [15] as

$$R_0 = \frac{a(k(1-\varrho) + \varrho k\omega)(1+\xi\mu)}{\xi\mu}.$$

3. Qualitative analysis

Qualitative theory of existence of solution to a dynamical system is an important consequence of the applied analysis, where we can get information about the problem weather it has a solution or not?. In this regards, fixed point theory is an important tool to be used to investigate the existence and uniqueness of solution to a dynamical system. Here, we use Banach and Schauder fixed point results to derive sufficient results in this respect.

Here, we can write right hand sides of the proposed model (3) as

$$\begin{cases} D_t^{\theta(x)}(\mathcal{S}(t)) = \mathscr{H}_1(t, \mathcal{S}(t), \mathcal{I}(t)), \ x, t \in [0, T], \\ D_t^{\theta(x)}(\mathcal{I}(t)) = \mathscr{H}_2(t, \mathcal{S}(t), \mathcal{I}(t)), \ x, t \in [0, T], \\ \mathcal{S}(0) = \mathcal{S}_0, \ \mathcal{I}(0) = \mathcal{I}_0. \end{cases} \tag{7}$$

Apply variable order integral $I_t^{\theta(x)}$ on both sides of (7) yields

$$\begin{cases} \mathcal{S}(t) = \mathcal{S}_0 + \dfrac{1}{\Gamma(\theta(x))} \displaystyle\int_0^t (t-\varsigma)^{\theta(x)-1} \mathscr{H}_1(\varsigma, \mathcal{S}(\varsigma), \mathcal{I}(\varsigma)) d\varsigma, \ x, t \in [0, T], \\ \mathcal{I}(t) = \mathcal{I}_0 + \dfrac{1}{\Gamma(\theta(x))} \displaystyle\int_0^t (t-\varsigma)^{\theta(x)-1} \mathscr{H}_2(\varsigma, \mathcal{S}(\varsigma), \mathcal{I}(\varsigma)) d\varsigma, \ x, t \in [0, T]. \end{cases} \tag{8}$$

In addition, if $0 \leq t \leq T < \infty$, then

$$\mathbf{E} = C([0,T] \times \mathbf{R}^2+, \mathbf{R}_+) \times C([0,T] \times \mathbf{R}^2+, \mathbf{R}_+)$$

is the Banach space with norm $\|(\mathcal{S},\mathcal{I})\| = \max_{t\in[0,T]} |\mathcal{S}(t)| + \sup_{t\in[0,T]} |\mathcal{I}(t)|$. Moreover, the given hypothesis hold:

(A1) For every $(\mathcal{S},\mathcal{I}), (\bar{\mathcal{S}},\bar{\mathcal{I}}) \in \mathbf{E}$, and there exist constants $\mathbf{L}_i(i=1,2)$, such that

$$|\mathscr{H}_i(t,\mathcal{S},\mathcal{I}) - \mathscr{H}_i(t,\bar{\mathcal{S}},\bar{\mathcal{I}})| \leq \mathbf{L}_i\Big[|\mathcal{S}-\bar{\mathcal{S}}| + |\mathcal{I}-\bar{\mathcal{I}}|\Big].$$

(A2) For every $(\mathcal{S},\mathcal{I}) \in \mathbf{E}$, there constants $\mathbf{C}_i,\ \mathbf{M}_i(i=1,2) > 0$, such that

$$|\mathscr{H}_i(t,\mathcal{S},\mathcal{I})| \leq \mathbf{C}_i\Big[|\mathcal{S}| + |\mathcal{I}|\Big] + \mathbf{M}_i.$$

Theorem 3.1. *Inview of hypothesis A1 and if the conditions $\frac{T^{\theta(x)}}{\Gamma(\theta(x)+1)}\mathbf{L} < 1$ holds, the model (3) has a unique solution.*

Proof. As $\mathscr{T} : \mathbf{E} \to \mathbf{E}$ defined by from (8) as

$$\mathscr{T}(\mathcal{S},\mathcal{I}) = (\mathscr{T}_1, \mathscr{T}_2)(\mathcal{S},\mathcal{I}),$$

such that

$$\mathscr{T}_1(\mathcal{S},\mathcal{I}) = \mathcal{S}_0 + \frac{1}{\Gamma(\theta(x))}\int_0^t (t-\varsigma)^{\theta(x)-1}\mathscr{H}_1(\varsigma,\mathcal{S}(\varsigma),\mathcal{I}(\varsigma))d\varsigma,\ \ x,t \in [0,T], \tag{9}$$

$$\mathscr{T}_2(\mathcal{S},\mathcal{I}) = \mathcal{I}_0 + \frac{1}{\Gamma(\theta(x))}\int_0^t (t-\varsigma)^{\theta(x)-1}\mathscr{H}_2(\varsigma,\mathcal{S}(\varsigma),\mathcal{I}(\varsigma))d\varsigma,\ \ x,t \in [0,T]. \tag{10}$$

Consider $(\mathcal{S},\mathcal{I}), (\bar{\mathcal{S}},\bar{\mathcal{I}}) \in \mathbf{E}$, then from first equation of (9), one has

$$\begin{aligned}\|\mathscr{T}_1(\mathcal{S},\mathcal{I}) - \mathscr{T}_1(\bar{\mathcal{S}},\bar{\mathcal{I}})\| &= \max_{t\in[0,T]}\Bigg|\frac{1}{\Gamma(\theta(x))}\int_0^t (t-\varsigma)^{\theta(x)-1}\mathscr{H}_1(\varsigma,\mathcal{S}(\varsigma),\mathcal{I}(\varsigma))d\varsigma \\ &\quad - \frac{1}{\Gamma(\theta(x))}\int_0^t (t-\varsigma)^{\theta(x)-1}\mathscr{H}_1(\varsigma,\bar{\mathcal{S}}(\varsigma),\bar{\mathcal{I}}(\varsigma))d\varsigma\Bigg| \\ &\leq \frac{\mathbf{L}_1 T^{\theta(x)}}{\Gamma(\theta(x)+1)}\Big[\|\mathcal{S}-\bar{\mathcal{S}}\| + \|\mathcal{I}-\bar{\mathcal{I}}\|\Big].\end{aligned} \tag{11}$$

In the same way from second equation of (9), one has

$$\|\mathscr{T}_2(\mathcal{S},\mathcal{I}) - \mathscr{T}_2(\bar{\mathcal{S}},\bar{\mathcal{I}})\| \leq \frac{\mathbf{L}_2 T^{\theta(x)}}{\Gamma(\theta(x)+1)}\Big[\|\mathcal{S}-\bar{\mathcal{S}}\| + \|\mathcal{I}-\bar{\mathcal{I}}\|\Big]. \tag{12}$$

Let $\mathbf{L}_1 + \mathbf{L}_2 = \mathbf{L}$, we have from (11), and (12)

$$\|\mathscr{T}(\mathcal{S},\mathcal{I}) - \mathscr{T}(\bar{\mathcal{S}},\bar{\mathcal{I}})\| \leq \frac{\mathbf{L}T^{\theta(x)}}{\Gamma(\theta(x)+1)}\|(\mathcal{S},\mathcal{I}) - (\bar{\mathcal{S}},\bar{\mathcal{I}})\|. \tag{13}$$

Since $\frac{\mathbf{L}T^{\theta(x)}}{\Gamma(\theta(x)+1)} < 1$, which yields that $\mathscr{T}$ is a contraction in (13). Thus the model (3) has a unique solution. □

Theorem 3.2. *Inview of hypothesis A2, the model (3) has at least one solution.*

Proof. Let

$$\mathcal{B} = \{(\mathcal{S},\mathcal{I}) \in \mathbf{E} :\ \ \|(\mathcal{S},\mathcal{I})\| \leq r\},$$

with $r \geq \frac{\mathcal{N}_0\Gamma(\theta(x)+1)+T^{\theta(x)}\mathbf{M}}{\Gamma(\theta(x)+1)-T^{\theta(x)}\mathbf{C}}$. We define an operator $\mathscr{A} : \mathcal{B} \to \mathcal{B}$ as

$$\mathscr{A}_1(\mathcal{S},\mathcal{I}) = \mathcal{S}_0 + \frac{1}{\Gamma(\theta(x))}\int_0^t (t-\varsigma)^{\theta(x)-1}\mathscr{H}_1(\varsigma,\mathcal{S}(\varsigma),\mathcal{I}(\varsigma))d\varsigma,\ \ x,t \in [0,T], \tag{14}$$

$$\mathscr{A}_2(\mathcal{S},\mathcal{I}) = \mathcal{I}_0 + \frac{1}{\Gamma(\theta(x))}\int_0^t (t-\varsigma)^{\theta(x)-1}\mathscr{H}_2(\varsigma,\mathcal{S}(\varsigma),\mathcal{I}(\varsigma))d\varsigma,\ \ x,t \in [0,T]. \tag{15}$$

Then for every $(\mathcal{S},\mathcal{I})\in\mathcal{B}$, one has

$$\begin{aligned}|\mathscr{A}_1(\mathcal{S},\mathcal{I})| &\le |\mathcal{S}_0|+\max_{t\in[0,T]}\left[\frac{1}{\Gamma(\theta(x))}\int_0^t (t-\varsigma)^{\theta(x)-1}|\mathscr{H}_1(\varsigma,\mathcal{S}(\varsigma),\mathcal{I}(\varsigma))|d\varsigma\right]\\ &\le |\mathcal{S}_0|+\max_{t\in[0,T]}\left[\frac{1}{\Gamma(\theta(x))}\int_0^t (t-\varsigma)^{\theta(x)-1}\left[\mathbf{C}_1[|\mathcal{S}(\varsigma)|+|\mathcal{I}(\varsigma)|]+\mathbf{M}_1\right]d\varsigma\right]\\ &\le |\mathcal{S}_0|+\frac{T^{\theta(x)}}{\Gamma(\theta(x)+1)}\left[\mathbf{C}_1[\|\mathcal{S}(\varsigma)\|+\|\mathcal{I}(\varsigma)\|]+\mathbf{M}_1\right],\end{aligned}\tag{16}$$

and also one has

$$|\mathscr{A}_1(\mathcal{S},\mathcal{I})|\le|\mathcal{I}_0|+\frac{T^{\theta(x)}}{\Gamma(\theta(x)+1)}\left[\mathbf{C}_2[\|\mathcal{S}(\varsigma)\|+\|\mathcal{I}(\varsigma)\|]+\mathbf{M}_2\right].\tag{17}$$

From (16) and (17), one has by using $\mathbf{C}_1+\mathbf{C}_2=\mathbf{C}$, $\mathbf{M}_1+\mathbf{M}_2=\mathbf{M}$,

$$\begin{aligned}\|\mathscr{A}_1(\mathcal{S},\mathcal{I})\| &\le \mathcal{N}_0+\frac{T^{\theta(x)}}{\Gamma(\theta(x)+1)}\left[\mathbf{C}[\|\mathcal{S}(\varsigma)\|+\|\mathcal{I}(\varsigma)\|]+\mathbf{M}\right]\\ &\le \mathcal{N}_0+\frac{T^{\theta(x)}}{\Gamma(\theta(x)+1)}\left[\mathbf{C}r+\mathbf{M}\right]\\ &\le r.\end{aligned}\tag{18}$$

Hence (18) yields that $(\mathcal{S},\mathcal{I})\in\mathcal{B}$. Therefore one has $\mathscr{A}(\mathcal{B})\subset\mathcal{B}$. Also It is obvious that $\mathcal{B}$ is bounded.

Let $t_m<t_n\in[0,T]$, then consider

$$\begin{aligned}|\mathscr{A}_1(\mathcal{S},\mathcal{I})(t_n)-\mathscr{A}_1(\mathcal{S},\mathcal{I})(t_m)| &= \left|\frac{1}{\Gamma(\theta(x))}\int_0^{t_n}(t_n-\varsigma)^{\theta(x)-1}\mathscr{H}_1(\varsigma,\mathcal{S}(\varsigma),\mathcal{I}(\varsigma))d\varsigma\right.\\ &\quad\left.-\frac{1}{\Gamma(\theta(x))}\int_0^{t_m}(t_m-\varsigma)^{\theta(x)-1}\mathscr{H}_1(\varsigma,\mathcal{S}(\varsigma),\mathcal{I}(\varsigma))d\varsigma\right|,\\ &\le \frac{1}{\Gamma(\theta(x))}\left[\int_0^{t_n}[(t_m-\varsigma)^{\theta(x)-1}-(t_n-\varsigma)^{\theta(x)-1}]|\mathscr{H}_1(\varsigma,\mathcal{S}(\varsigma),\mathcal{I}(\varsigma))|d\varsigma\right.\\ &\quad\left.+\int_{t_m}^{t_n}(t_n-\varsigma)^{\theta(x)-1}|\mathscr{H}_1(\varsigma,\mathcal{S}(\varsigma),\mathcal{I}(\varsigma))|d\varsigma\right],\\ &\le \frac{(\mathbf{C}_1r+\mathbf{M}_1)}{\Gamma(\theta(x)+1)}\left[t_n^{\theta(x)}-t_m^{\theta(x)}+2(t_n-t_m)^{\theta(x)}\right].\end{aligned}\tag{19}$$

As $t_n\to t_m$, then right side goes to zero in (19). Also, $\mathscr{A}_1$ is bounded and continuous. Therefore,

$$\|\mathscr{A}_1(\mathcal{S},\mathcal{I})(t_n)-\mathscr{A}_1(\mathcal{S},\mathcal{I})(t_m)\|\to 0,\ \text{as } t_n\to t_m.$$

In the same way, we can show for $\mathscr{A}_2$ as

$$\|\mathscr{A}_2(\mathcal{S},\mathcal{I})(t_n)-\mathscr{A}_2(\mathcal{S},\mathcal{I})(t_m)\|\to 0,\ \text{as } t_n\to t_m.$$

Therefore, we can say that

$$\|\mathscr{A}(\mathcal{S},\mathcal{I})(t_n)-\mathscr{A}(\mathcal{S},\mathcal{I})(t_n)\|\to 0,\ \text{as } t_n\to t_m.$$

Thus $\mathscr{A}$ is completely continues function, consequently, model (3) has at least one solution. □

4. Numerical solution

Here, we develop the numerical scheme for the considered model (3). We use Adams–Bashforth-Moulton Method [36]. Therefore, we consider a fractional order problem as

$$\begin{cases}\mathrm{D}_t^{\theta(x)}\mathcal{X}(t)=\Phi(t,\mathcal{X}(t)),\ x,t\in[0,T],\\ \mathcal{X}(0)=\mathcal{X}_0,\end{cases}\tag{20}$$

where $\Phi:[0,T]\times\mathbf{R}\to\mathbf{R}$. We can obtain the solution of (20) as

$$\mathcal{X}(t)=\mathcal{X}_0+\int_0^t\frac{(t-\varsigma)^{\theta(x)-1}}{\Gamma(\theta(x))}\Phi(\varsigma,\mathcal{X}(\varsigma))d\varsigma,\ x,t\in[0,T].\tag{21}$$

Table 1
Nomenclatures and numerical values [15].

Nomenclature	Numerical value	Nomenclature	Numerical value
a	0.00009	ϱ	45%, 70%, 90%
ω	0.00078	μ	0.019
ξ	100	k	0.0009

We use the discretization for (21) as $\mathrm{h} = \frac{T}{n}$, $t_n = n\mathrm{h}$, with $n = 0, 1, 2, \ldots$, we have

$$\mathcal{X}_{\mathrm{h}}(t_{n+1}) = \mathcal{X}_0 + \frac{\mathrm{h}^{\theta(x)}}{\Gamma(\theta(x)+2)}\Phi(t_{n+1}, \mathcal{X}_{\mathrm{h}}^{\mathrm{p}}(t_{n+1})) + \sum_{j=0}^{n}\frac{\mathrm{h}^{\theta(x)}\Lambda_{j,\ n+1}}{\Gamma(\theta(x)+2)}\Phi(t_j, \mathcal{X}_{\mathrm{h}}(t_j)), \tag{22}$$

where

$$\mathcal{X}_{\mathrm{h}}^{\mathrm{p}}(t_{n+1}) = \mathcal{X}_0 + \frac{1}{\Gamma(\theta(x))}\sum_{j=0}^{n}\Delta_{j,\ n+1}\Phi(t_j, \mathcal{X}_{\mathrm{h}}(t_j)), \tag{23}$$

The coefficients $\Lambda_{j,\ n+1}$, $\Delta_{j,\ n+1}$ in (22), and (23) are described as

$$\Lambda_{j,\ n+1} = \begin{cases} n^{theta(x)+1} - (n-\theta(x))(n+1)^{\theta(x)}, & j = 0, \\ (n-j-2)^{\theta(x)+1} + (n-j)^{\theta(x)+1} - 2(n-j+1)^{\theta(x)+1}, & 1 \le j \le n, \\ 1, & j = n+1, \end{cases} \tag{24}$$

and

$$\Delta_{j,\ n+1} = \frac{h^{\theta(x)}}{\theta(x)}\left((n-j+1)^{\theta(x)} - (n-j)^{\theta(x)}\right). \tag{25}$$

The error estimate can be computed as done in [37]

$$\max_{j=0,1,2,\ldots,N}\left|\mathcal{X}(t_j) - \mathcal{X}_{\mathrm{h}}(t_j)\right| = O(\mathrm{h}^{\theta}(x)),\ \ x \in [0, 1],$$

such that $0 < \theta(x) \le 1$.

Now inview of the above formula for numerical investigation, we can establish the desired numerical scheme for the proposed model (3) as

$$\mathcal{S}_{\mathrm{h}}(t_{n+1}) = \mathcal{S}_0 + \frac{\mathrm{h}^{\theta(x)}}{\Gamma(\theta(x)+2)}\mathscr{H}_1(t_{n+1}, \mathcal{S}_{\mathrm{h}}^{\mathrm{p}}(t_{n+1}), \mathcal{I}_{\mathrm{h}}^{\mathrm{p}}(t_{n+1})) + \sum_{j=0}^{n}\frac{\mathrm{h}^{\theta(x)}\Lambda_{j,\ n+1}}{\Gamma(\theta(x)+2)}\mathscr{H}_1(t_j, \mathcal{S}_{\mathrm{h}}(t_j), \mathcal{I}_{\mathrm{h}}(t_j)), \tag{26}$$

where

$$\mathcal{S}_{\mathrm{h}}^{\mathrm{p}}(t_{n+1}) = \mathcal{S}_0 + \frac{1}{\Gamma(\theta(x))}\sum_{j=0}^{n}\Delta_{j,\ n+1}\mathscr{H}_1(t_j, \mathcal{S}_{\mathrm{h}}(t_j), \mathcal{I}_{\mathrm{h}}(t_j)), \tag{27}$$

and in the same way, for the other compartment one has

$$\mathcal{I}_{\mathrm{h}}(t_{n+1}) = \mathcal{I}_0 + \frac{\mathrm{h}^{\theta(x)}}{\Gamma(\theta(x)+2)}\mathscr{H}_2(t_{n+1}, \mathcal{S}_{\mathrm{h}}^{\mathrm{p}}(t_{n+1}), \mathcal{I}_{\mathrm{h}}^{\mathrm{p}}(t_{n+1})) + \sum_{j=0}^{n}\frac{\mathrm{h}^{\theta(x)}\Lambda_{j,\ n+1}}{\Gamma(\theta(x)+2)}\mathscr{H}_2(t_j, \mathcal{S}_{\mathrm{h}}(t_j), \mathcal{I}_{\mathrm{h}}(t_j)), \tag{28}$$

where

$$\mathcal{I}_{\mathrm{h}}^{\mathrm{p}}(t_{n+1}) = \mathcal{I}_0 + \frac{1}{\Gamma(\theta(x))}\sum_{j=0}^{n}\Delta_{j,\ n+1}\mathscr{H}_2(t_j, \mathcal{S}_{\mathrm{h}}(t_j), \mathcal{I}_{\mathrm{h}}(t_j)). \tag{29}$$

With the help of the above relations (26), and (28), we simulate the results of our propped model by using various variable orders.

5. Numerical results and discussion

To simulate the variable order model (3), we utilize the aforesaid scheme established in (26), and (28). For this need, we use different variable orders and various values for isolation effects. Therefore, in this connection, we considered some real data of Pakistan from [38] about infected cases in the given Table 1. Let the total population of the country be approximately equal to $\mathcal{N} = 220.142$ millions, $\mathcal{S}_0 = 218.563642$ millions, and $\mathcal{I}_0 = 1.578358$ millions.

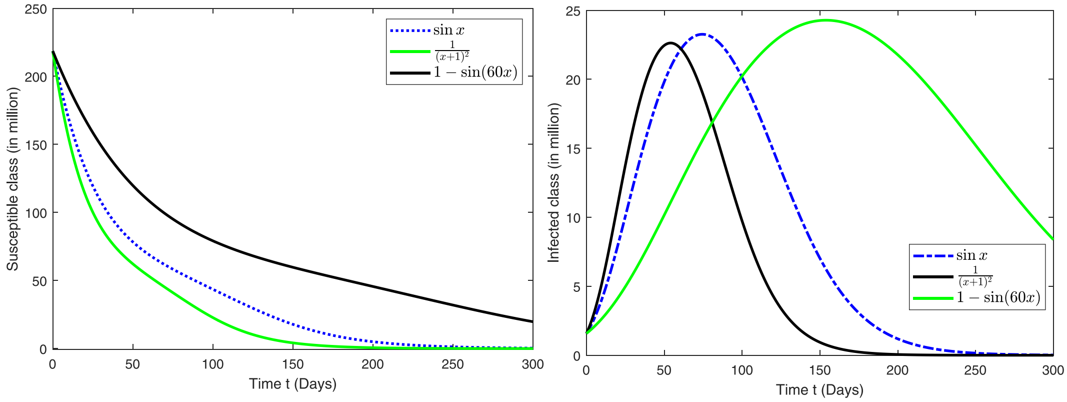

Fig. 1. Numerical simulations of both compartments of the model (3) for the given three different variable order with $\varrho = 45\%$.

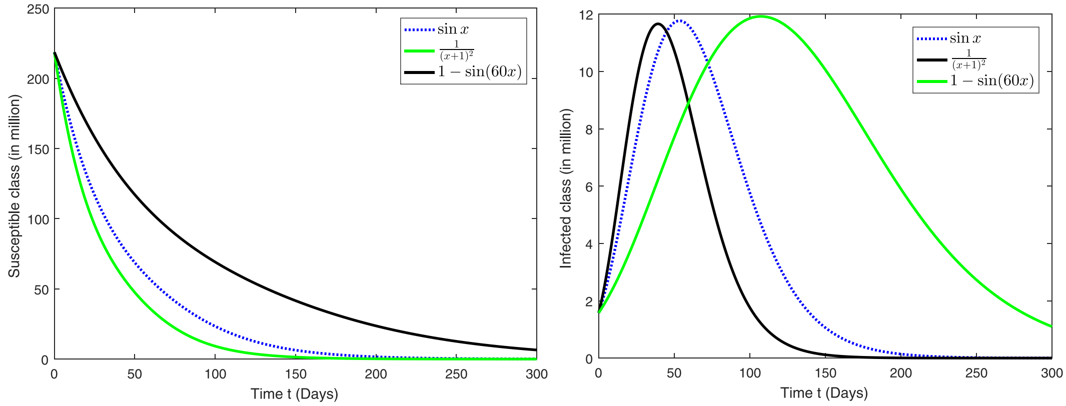

Fig. 2. Numerical simulations of both compartments of the model (3) for the given three different variable order with $\varrho = 50\%$.

Case-I, when $\varrho = 0.45$: On using the numerical values given above in Table 1, we have presented the numerical results of both compartments using various variable order for $\theta(x)$ as shown in Fig. 1. From Fig. 1, we observe that at the contact (isolation) rate of 45% and using three different variable orders that is $\theta(x) = \sin x, \frac{1}{(x+1)^2}, 1 - \sin(60x)$, the susceptible class shows decay in its behaviors, because the infection is very rapidly increasing during the first few months. The concerned decline and growth have shown at different variable orders. If the value of function $\theta \to 1$, we can get the dynamical behavior like at integer order 1. We use data of [39].

Case-II, when $\varrho = 0.50$: From Fig. 2, we observe that at the isolation if increase up to 50% and using the same three different variable orders as $\theta(x) = \sin x, \frac{1}{(x+1)^2}, 1 - \sin(60x)$, the susceptible class shows decay in its behaviors, because the infection is very rapidly increasing for very less interval of months. The concerned decline and growth are different at different variable orders.

Case-III, when $\varrho = 0.60$ From Fig. 3, we observe that at the isolation if increase up to 60% and using the same three different variable orders as $\theta(x) = \sin x, \frac{1}{(x+1)^2}, 1 - \sin(60x)$, the susceptible class shows decay in its behaviors slight slow, while the infection increases for less interval of months. The concerned decline and growth are different at different variable orders also. From the above mentioned figures, we see that biologically when protection is low that people do not take care of precautionary measures, then the population of uninfected people will go on decaying. As a result the infection population will increase exponentially. In addition, more the protection or following the precautionary measures, less will be the transmission of infection cases. Moreover, here we compare the simulated results of infected class with real data for Pakistan for 180 days from 1 September 2020 to 27the February 2021 (see [38]) in Fig. 4. We see that the graphical results at $\theta(x) = 1 - \sin x$ and real data are closely agree.

6. Conclusion

This manuscript aims to attempt on variable order $\mathcal{SI}$ dynamical system for COVID-19. Since the variable order differentiations and integrations are the natural extension of classical as well as fractional order integral and differential operators. Here, we have considered the variable order θ is a continuous function of $x \in [0, T]$ such that $\theta(x) \in (0, 1]$.

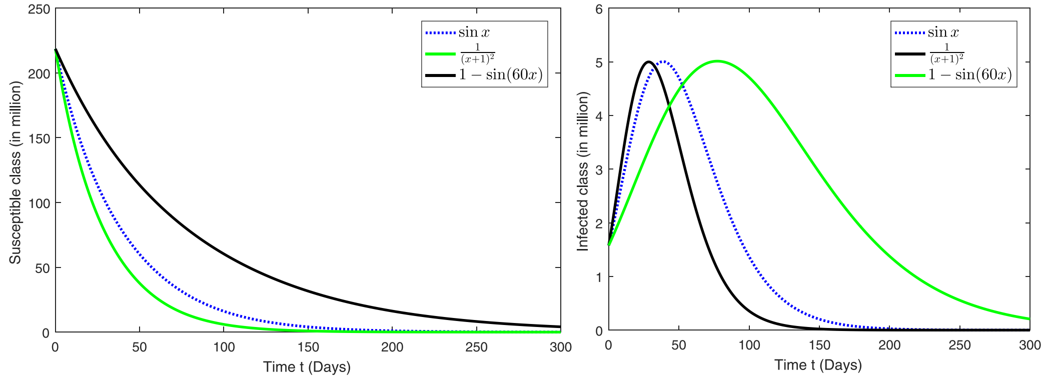

Fig. 3. Numerical simulations of both compartments of the model (3) for the given three different variable order with $\varrho = 60\%$.

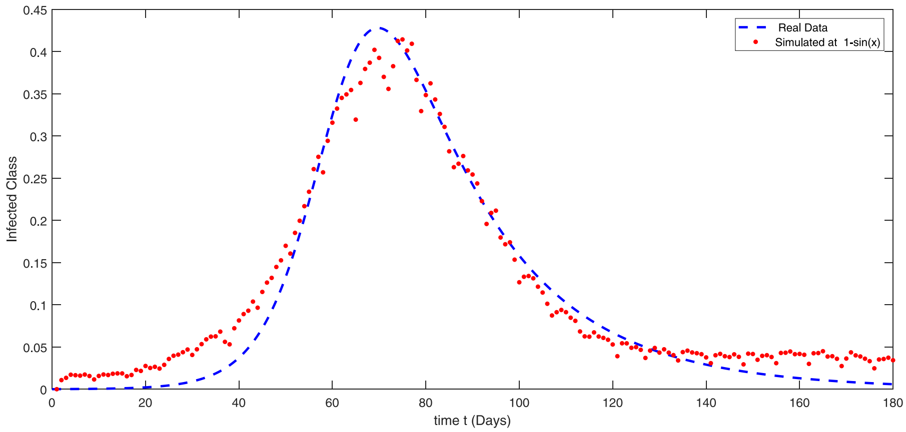

Fig. 4. Comparison between simulated data and numerical result at $\theta(x) = 1 - \sin x$.

Further, existence and uniqueness results have been established with the help of some fixed point results. Further an numerical scheme has also established to simulate the results. Moreover, we have simulated the model by using some real values corresponding to different variable orders for three different cases. We have testified our model for three different isolation values in percentage and the effect of isolation has been observed in three Figs. 1–3. In addition, we have also given a comparison between real and simulated data. We have observed that both results agree very at $\theta(x) = 1 - \sin x$. Hence we conclude that variable order derivatives and integrals also play a vital roles in numerical and theoretical investigations of various dynamical problems. Here, we remark that for proper comparison a suitable function should be choose to compare real and simulated results.

Declaration of competing interest

Does not exist any conflict of interest.

Data availability

No data was used for the research described in the article

Acknowledgment

Authors have read and approved the final version.

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